The gravitational self-force correction to the binding energy of compact binaries

Enrico Barausse
(University of Guelph &
CITA National Fellow)

in collaboration with
Alexandre Le Tiec &
Alessandra Buonanno

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Outline

- The two-body problem in general relativity
- → Why is it important?
- → How do we solve it:
  - Post-Newtonian theory (PN)
  - Numerical relativity (NR)
  - The gravitational self-force (SF)
- The binding energy of compact binaries at leading order beyond test-particle limit
- Applications:
  - The ISCO shift due to the SF
  - Comparison to NR
  - Calculation of PN binding energy through 6PN order
  - The exact effective-one-body metric at linear order in the mass ratio

Why WAS the two-body problem important?

- Historically “the” problem in gravity

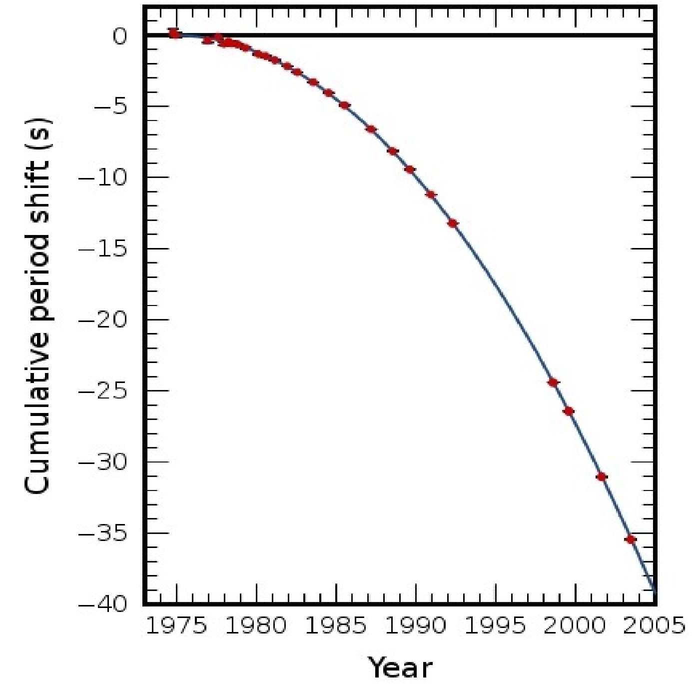
  - Aristotle, Copernicus, Galileo, Kepler, Newton...

  - In GR, dates back to Einstein (Mercury's perihelion precession, 1916) and Schwarzschild (1916)

- More recently, basis for experimental tests of GR

  - Solar system tests

  - Hulse-Taylor pulsar



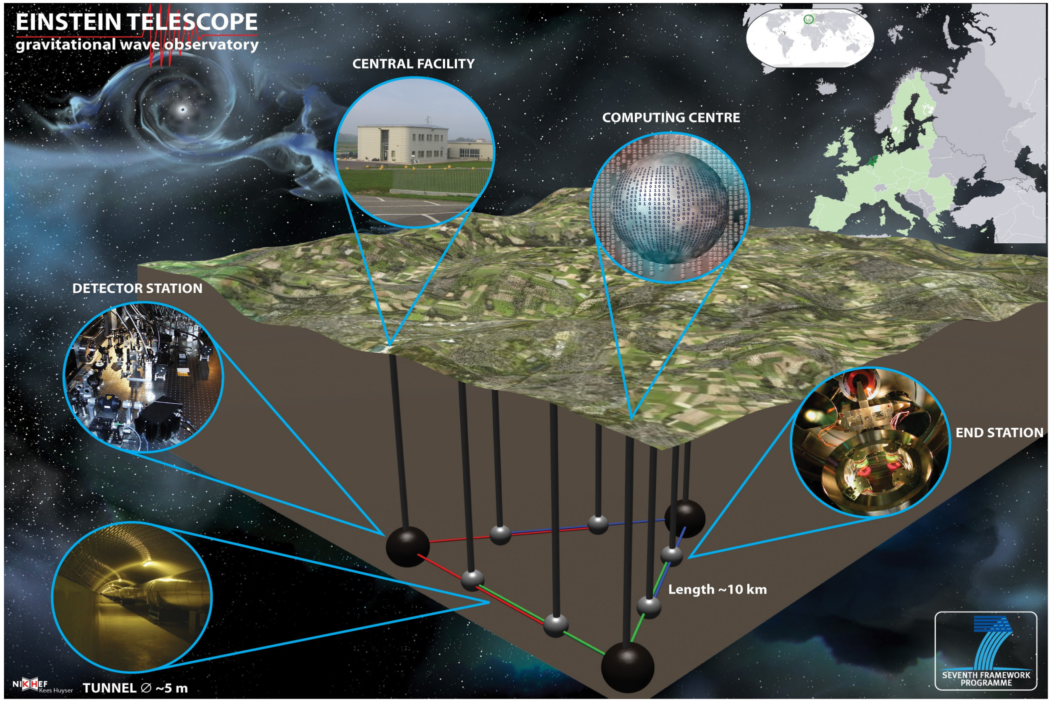
Why IS the two-body problem STILL important?

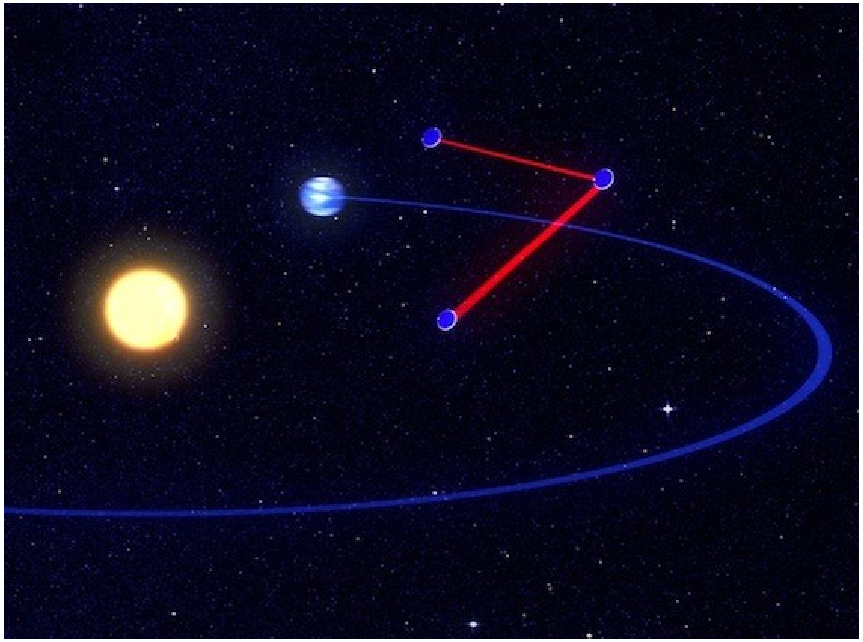
Gravitational waves!

Existing detectors...

Why IS the two-body problem STILL important?

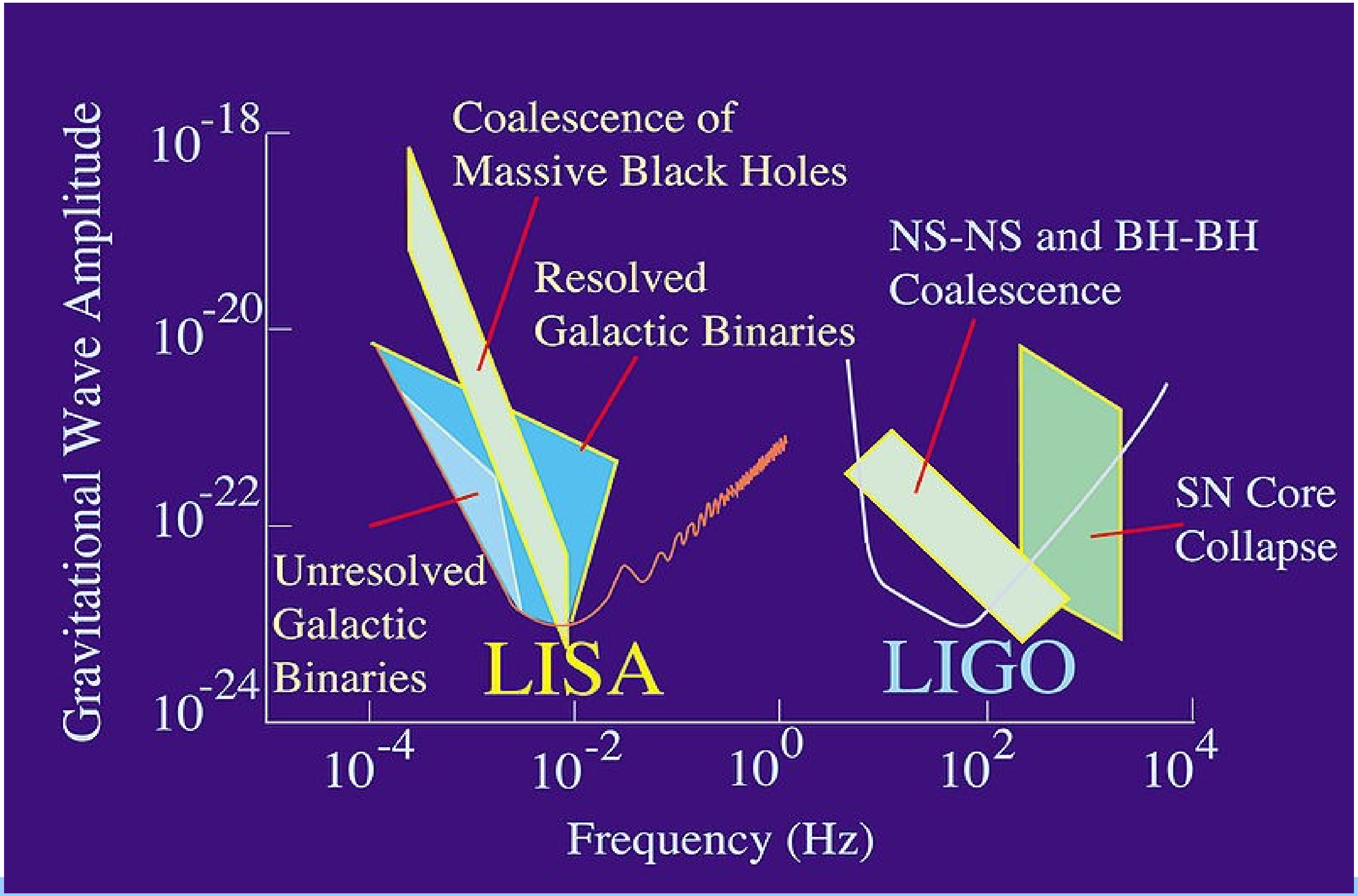
… and future ones!





Why IS the two-body problem STILL important?

Sources for GW detectors are compact binaries



How to solve the two-body problem

- Solvable analytically in Newtonian theory and for a test-particle particle in Schwarzschild/Kerr

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- Test particle (geodesic motion) = GR at lowest order in mass ratio m_{part}/m_{schw} (m_{part}/m_{kerr})
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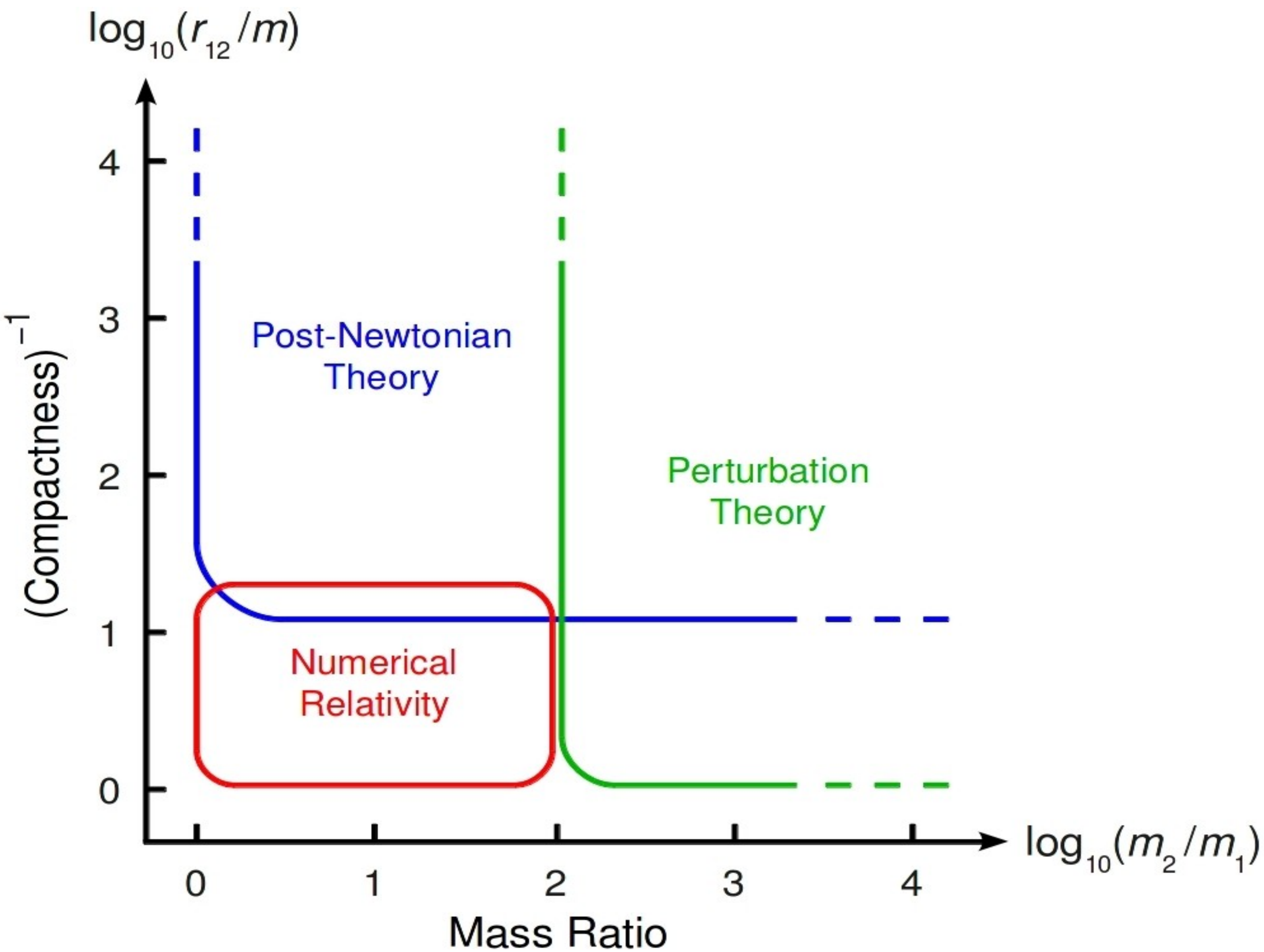
- Test particle (geodesic motion) = GR at lowest order in mass ratio m_{part}/m_{schw} (m_{part}/m_{kerr})
  → two-body problem solved perturbatively in mass ratio (SF formalism)
- NR simulations can solve for BH binaries (with spins) for mass ratios $\gtrsim 1/100$

The two-body problem in GR

Techniques have different ranges of validity



PN theory: the results

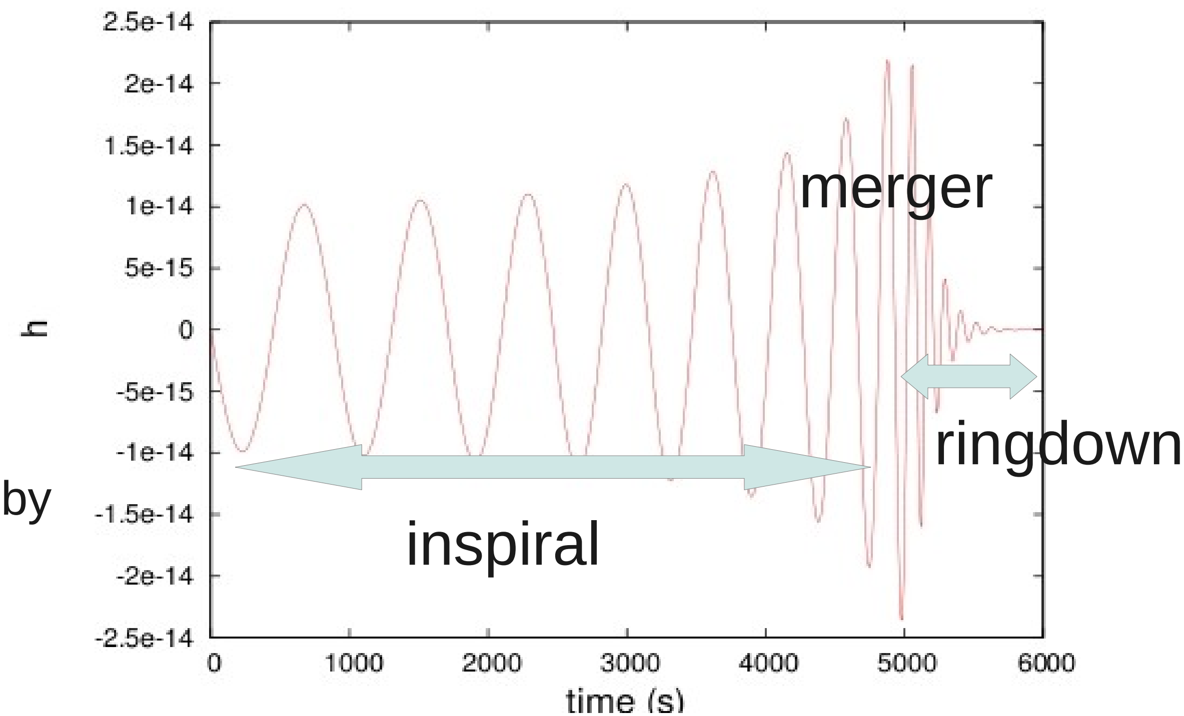
- Conservative dynamics (Hamiltonian, binding energy, angular momentum, frequency, ISCO, etc) through 3PN for non-spinning binaries and 3.5PN order for spinning ones

- Dissipative dynamics (E and L fluxes) through 3/3.5PN for non-spinning binaries and 3PN order for spinning ones

- Waveforms through 3PN for non-spinning binaries and 2.5PN order for spinning ones

Numerical Relativity: the results

- Gravitational waveforms (Pretorius 2005, Campanelli et al 2006, Baker et al 2006)



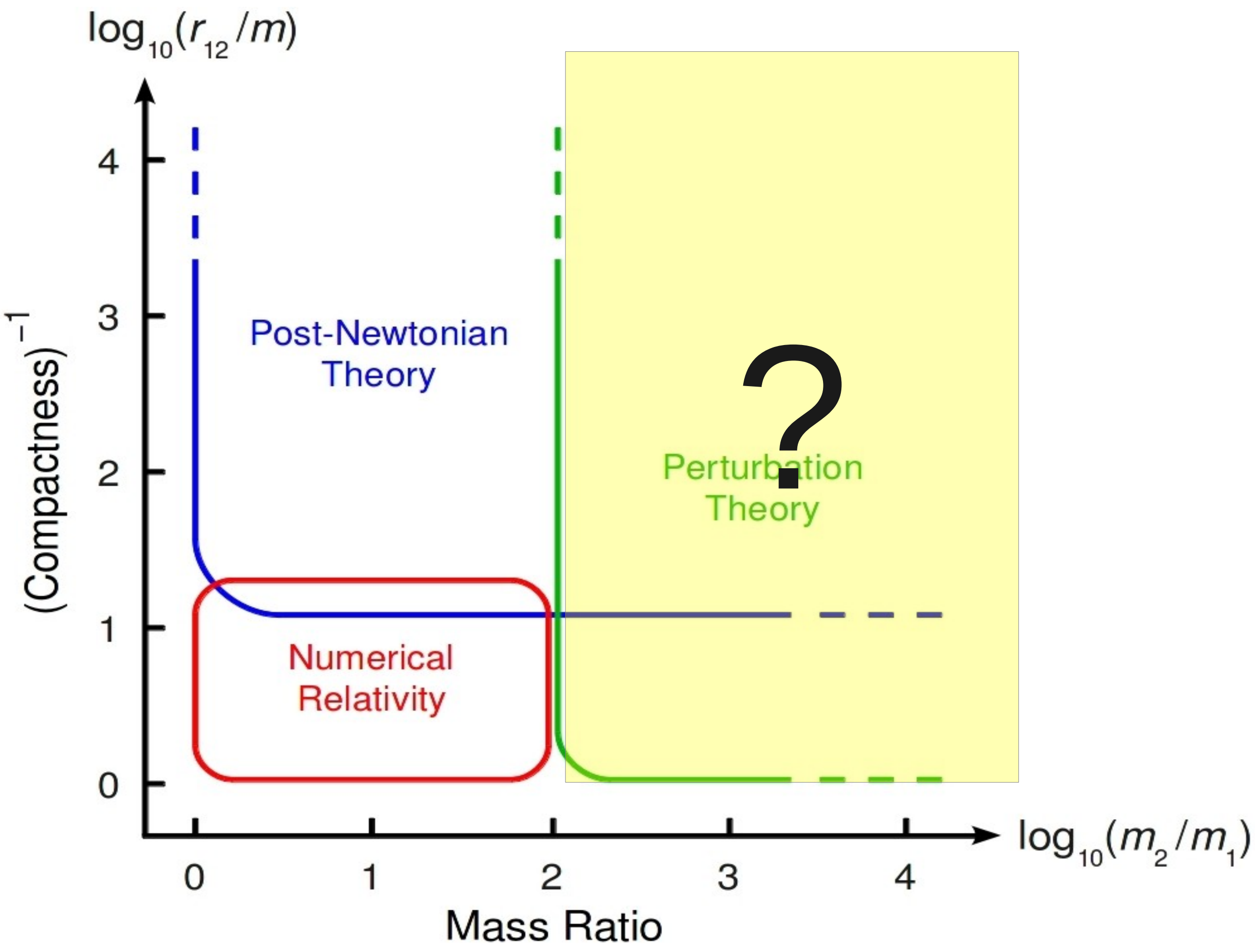
(Waveform produced by the Golm AEI group)

- Final mass, spin and kick velocity of BH remnant

It takes weeks/months to generate NR waveforms: too slow for data analysis!

The two-body problem in GR

Techniques have different ranges of validity



The gravitational self-force in a nutshell

Motion of small BH with mass m_1 in Schwarzschild background with mass m_2

- Near BH, $g=g_{BH}+O(r/m_2)+O(r/m_2)^2$
- Far away, $g=g_{bkgd}+O(m_1/m_2)+O(m_1/m_2)^2$
- Matching in a buffer region where both pictures are valid, one finds the BH's eqs of motion

$$u^\mu \nabla_\mu u^\nu = f^\nu_{cons} + f^\nu_{diss}$$

$f^\nu_{cons}, f^\nu_{diss} = O\left(m_1/m_2\right)$ are the SF

Derived for BH, but result valid also for classical ”particle”

Physical meaning of the SF

$$u^{\mu}\nabla_{\mu}u^{\nu}=f^{\nu}_{cons}+f^{\nu}_{diss}\sim\nabla h^{reg}$$

$h^{reg}=O(m_1/m_2)$ = perturbation produced by particle, but regularized to avoid divergence at particle's position

SF = interaction of particle with itself!

$$\tilde{u}^{\mu}\tilde{\nabla}_{\mu}\tilde{u}^{\nu}=0, \quad \tilde{g}=g+h^{reg} \; ,$$

Particle moves on geodesic of ”perturbed” metric

The gravitational self-force: the results

More meager record because technically challenging (+ sociological reasons):

- Calculation of the SF-induced ISCO shift for a particle in Schwarzschild (Barack & Sago 2009)

$$m\Omega_{\rm ISCO} = 6^{-3/2}\left[1+\nu C_\Omega + \mathscr{O}(\nu^2)\right]$$

$$m = m_1 + m_2, \quad \nu = \frac{m_1 m_2}{m^2} = \frac{m_1}{m_2} + O\left(\frac{m_1}{m_2}\right)^2, \quad C_\Omega = 1.2512(4)$$

- Effect of SF on periastron precession (Barack and Sago 2009)
- First waveforms being computed (Warburton et al 2011, Wardell et al 2011)

Another result: the redshift (Detweiler 2008)

- Spacetime of binary that remains on circular orbit forever has helical Killing vector (reasonable approx to real binaries in adiabatic regime)

$$K = \partial_t + \Omega \partial_\phi$$

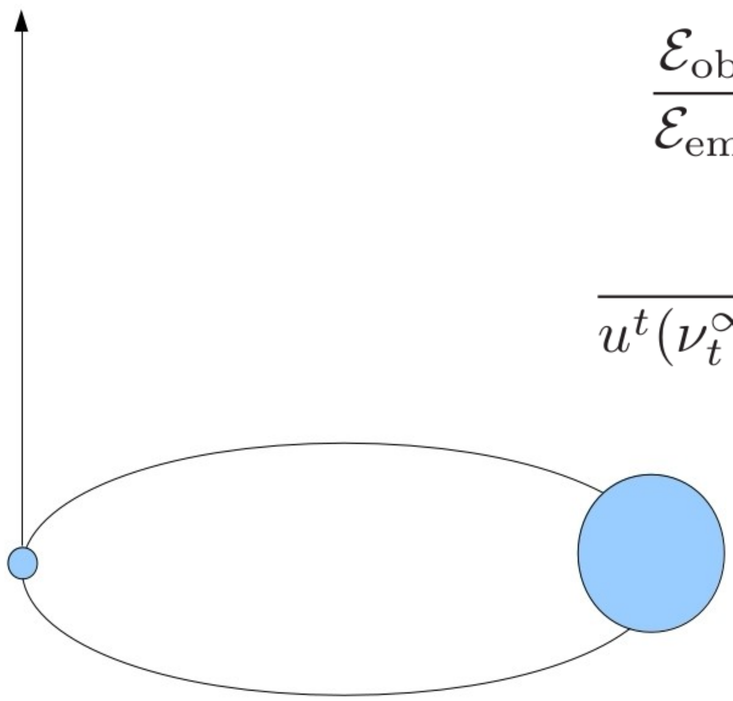
- Effect of the SF on the projection of the 4-velocity of a circular orbit in Schwarzschild on helical killing vector calculated with high accuracy (Detweiler 2008)

$$z \equiv -u \cdot K = \sqrt{1-3x} + \nu z_{SF}(x) + O(\nu)^2$$

$$x = (m\Omega)^{2/3}$$

Another result: the redshift (Detweiler 2008)

Can be interpreted as redshift of photon emitted by particle along z axis → "redshift" observable



$$\frac{\mathcal{E}_{\rm ob}}{\mathcal{E}_{\rm em}} = \frac{u^a_{\rm ob}\nu_a}{u^a_{\rm em}\nu_a} = \frac{u^t_\infty \nu^\infty_t}{u^t (k^a\nu_a)_{\rm em}} = \frac{\nu^\infty_t}{u^t (k^a\nu_a)_\infty}$$

$$\frac{\nu^\infty_t}{u^t(\nu^\infty_t + \Omega\nu^\infty_\phi)} = \frac{1}{u^t} - \frac{\Omega\nu^\infty_\phi}{u^t(\nu^\infty_t + \Omega\nu^\infty_\phi)} = \frac{1}{u^t}$$

$$u = u^t \partial_t + u^\phi \partial_\phi = u^t K$$

$$-1 = u\cdot u = (u^t)^2 K\cdot K$$

$$z = -u\cdot K = 1/u^t$$

A new meaning for the redshift observable?

- Must be related to binary's binding energy: circular binaries fully described by total ADM mass M(Ω), because $\frac{\partial M}{\partial \Omega} = \frac{\partial J}{\partial \Omega} \Omega$

  (this comes from Hamilton eqs $\dot{\phi} = \Omega = \frac{\partial H}{\partial p_\phi} = \frac{\partial H}{\partial J}$)

- For isolated BH, $\frac{\partial M}{\partial J} = \Omega_H$ if A = constant

$$\delta M - \Omega_{\rm H}\, \delta J = \kappa\, \delta A/(8\pi) = (4 m_{\rm irr}\, \kappa)\, \delta m_{\rm irr}$$

$$m_{\rm irr}^2 = A/(16\pi)$$

Can we write a 1st thermodynamics law for binaries?

A generalized first law of BH thermodynamics
(Friedman, Urȳ and Shibata 2004)

- In spacetime with a perfect fluid and global helical symmetry, variations of helical charge related to variations of baryonic mass, entropy, and vorticity

$$\delta M - \Omega \delta J = \int_\Sigma \left[\bar{\mu}\,\Delta(\mathrm{d}M_\mathrm{b}) + \bar{T}\,\Delta(\mathrm{d}S) + v^\alpha \Delta(\mathrm{d}C_\alpha)\right] + \sum_n \frac{\kappa_n}{8\pi}\,\delta A_n$$

$$\bar{T} = T/u^t \qquad \mathrm{d}M_\mathrm{b} \equiv \rho u^\alpha \mathrm{d}\Sigma_\alpha \,,$$

$$h = (\varepsilon + P)/\rho \qquad \mathrm{d}S \equiv s\,\mathrm{d}M_\mathrm{b} \,,$$

$$\bar{\mu} = (h - Ts)/u^t \qquad \mathrm{d}C_\alpha \equiv h u_\alpha \mathrm{d}M_\mathrm{b} \,,$$

$$u^\alpha = u^t(K^\alpha + v^\alpha)$$

- But definition of M and J only possible in asymptotically flat spacetimes...

Helical spacetimes cannot be asymptotically flat

- Intuitively: if binary remains on circular orbit forever, GWs need to enter the system to balance binary's own emission → far from binaries, standing waves dominate energy content of spacetime

- Asymptotic flatness can be recovered if GW emission can be “turned off”: possible in approximations to GR

  - conformal flatness approximation

  - PN theory and SF formalism (conservative and dissipative dynamics unambiguously disentangled)

Specialize 1st law to binaries of point particles

- Under asymptotic flatness assumption, 1st law becomes

$$\delta M - \Omega\delta J = \int_\Sigma \left[\bar{\mu}\,\Delta(\mathrm{d}M_\mathrm{b}) + \bar{T}\,\Delta(\mathrm{d}S) + v^\alpha\Delta(\mathrm{d}C_\alpha)\right] + \sum_n \frac{\kappa_n}{8\pi}\,\delta A_n$$

$$\bar{T} = T/u^t \qquad h = (\varepsilon + P)/\rho \qquad \mathrm{d}M_\mathrm{b} \equiv \rho u^\alpha \mathrm{d}\Sigma_\alpha\,,$$

$$\bar{\mu} = (h - Ts)/u^t \qquad u^\alpha = u^t(K^\alpha + v^\alpha) \qquad \mathrm{d}S \equiv s\,\mathrm{d}M_\mathrm{b}\,,$$

$$\mathrm{d}C_\alpha \equiv h u_\alpha \mathrm{d}M_\mathrm{b}\,,$$

- Stress-energy tensor of binary of point particles = perfect fluid with p=0 and

$$\rho(\boldsymbol{x},t) = \frac{1}{\sqrt{-g}}\sum_{A=1}^{2} m_A\, z_A\, \delta[\boldsymbol{x} - \boldsymbol{y}_A(t)]$$

- For this “fluid”, T=0, v^α=0, h=1, $\bar{\mu} = z$ and $\mathrm{d}M_\mathrm{b} = \mathrm{d}^3x \sum_{A=1}^{2} m_A\ \delta(\boldsymbol{x} - \boldsymbol{y}_A)$

$$\delta M - \Omega\,\delta J = z_1\,\delta m_1 + z_2\,\delta m_2$$

1st law of particle mechanics

$$\delta M - \Omega\, \delta J = z_1\, \delta m_1 + z_2\, \delta m_2$$

- Relation verified through 3PN order in PN theory (Le Tiec, Blanchet, Detweiler 2012)
- M must be homogenous of order 1 in $J^{1/2}$, m_1 and m_2 (because all these quantities have dimensions of a mass if G=c=1) → Euler theorem gives

$$M = J^{1/2} \frac{\partial M}{\partial J^{1/2}} + m_1 \frac{\partial M}{\partial m_1} + m_2 \frac{\partial M}{\partial m_2}$$

$$\Rightarrow \quad M - 2\Omega J = m_1 z_1 + m_2 z_2$$

- This "integrated" 1st law can be proved directly using definition of Komar charge of helical symmetry and Einstein equations

The 1st law and the redshift observable

$$\delta M - \Omega\,\delta J = z_1\,\delta m_1 + z_2\,\delta m_2$$

$$\rightarrow \quad \partial M/\partial m_A - \Omega\,\partial J/\partial m_A = z_A$$

$$\partial M/\partial \Omega = \Omega\,\partial J/\partial \Omega$$

Change of variables $(\Omega, m_1, m_2) \rightarrow (m, \nu, x)$

$$\nu \equiv m_1 m_2/m^2 \equiv \mu/m \qquad x \equiv (m\Omega)^{2/3} \qquad m = m_1 + m_2\,,$$

$$m z_1 = \mathcal{M} + \frac{2x}{3}\frac{\partial \mathcal{M}}{\partial x} + \frac{1-4\nu+\Delta}{2}\frac{\partial \mathcal{M}}{\partial \nu} \qquad \mathcal{M} \equiv M - \Omega J$$

$$M = \mathcal{M} - \frac{2x}{3}\frac{\partial \mathcal{M}}{\partial x}\,, \quad J = -\frac{2m}{3\sqrt{x}}\frac{\partial \mathcal{M}}{\partial x} \qquad \Delta \equiv (m_2 - m_1)/m = \sqrt{1-4\nu}$$

The 1st law and the redshift observable

- At first order beyond test-particle limit (i.e. at the "SF order"), define

$$z_1 = \sqrt{1-3x} + \nu\, z_{\rm SF}(x) + \mathscr{O}(\nu^2)$$

$$\hat{E} \equiv (M-m)/\mu = \left(\frac{1-2x}{\sqrt{1-3x}} - 1\right) + \nu\, E_{\rm SF}(x) + \mathscr{O}(\nu^2)$$

$$\hat{J} \equiv J/(m\mu) = \frac{1}{\sqrt{x(1-3x)}} + \nu\, J_{\rm SF}(x) + \mathscr{O}(\nu^2)$$

$$\hat{\mathscr{M}} \equiv (\mathscr{M} - m)/\mu = (\sqrt{1-3x} - 1) + \nu\, \mathscr{M}_{\rm SF}(x) + \mathscr{O}(\nu^2)$$

- Replacing in previous expressions, we get

$$E_{\rm SF}(x) = \mathscr{M}_{\rm SF}(x) - \tfrac{2x}{3}\mathscr{M}'_{\rm SF}(x) \qquad J_{\rm SF}(x) = -\tfrac{2}{3\sqrt{x}}\mathscr{M}'_{\rm SF}(x)$$

$$z_{\rm SF}(x) = 2\mathscr{M}_{\rm SF}(x) - 2(\sqrt{1-3x} - 1) - x/\sqrt{1-3x},$$

- … and eliminating $\mathscr{M}_{\rm SF}$ we'll have SF contribution to binding energy!

The SF contribution to the binding energy of compact binaries

$$E_{\rm SF}(x) = \frac{1}{2} z_{\rm SF}(x) - \frac{x}{3} z'_{\rm SF}(x) - 1 + \sqrt{1-3x} + \frac{x}{6} \frac{7-24x}{(1-3x)^{3/2}} ,$$

$$J_{\rm SF}(x) = -\frac{1}{3\sqrt{x}} z'_{\rm SF}(x) + \frac{1}{6\sqrt{x}} \frac{4-15x}{(1-3x)^{3/2}} .$$

- $z_{SF}(x)$ calculated numerically for $x \le 1/5$ ($r \gtrsim 5m$) with accuracy 10^{-6} (Detweiler 2008)
- Data fitted to within 10^{-5} by

$$z_{\rm SF}(x) = 2x\,(1 + a_1 x + a_2 x^2)/(1 + a_3 x + a_4 x^2 + a_5 x^3)$$

$$a_1 = -2.18522,\ a_2 = 1.05185,$$

$$a_3 = -2.43395,\ a_4 = 0.400665,\ \text{and}\ a_5 = -5.9991$$

The SF contribution to the binding energy of compact binaries

Result fully describes conservative dynamics of binaries at leading order in mass ratio beyond test-particle limit

Applications:

- Calculation of the ISCO shift due to SF effects (Le Tiec, Barausse and Buonanno 2012)

- Comparison to NR and PN results for comparable-mass binaries (Le Tiec, Barausse and Buonanno 2012)

- Derive unknown terms in the PN binding energy (Le Tiec, Blanchet and Detweiler 2012)

- Derive the **exact** effective-one-body metric at linear order in the mass ratio (Barausse, Le Tiec and Buonanno 2012)

The ISCO shift

- Barack and Sago (2009) found

$$m\Omega_{\rm ISCO} = 6^{-3/2}\left[1 + \nu C_\Omega + \mathcal{O}(\nu^2)\right] \qquad C_\Omega = 1.2512(4)$$

- If motion described by a Hamiltonian, ISCO coincides with the minimum of the binding energy

  Looking for this minimum we find

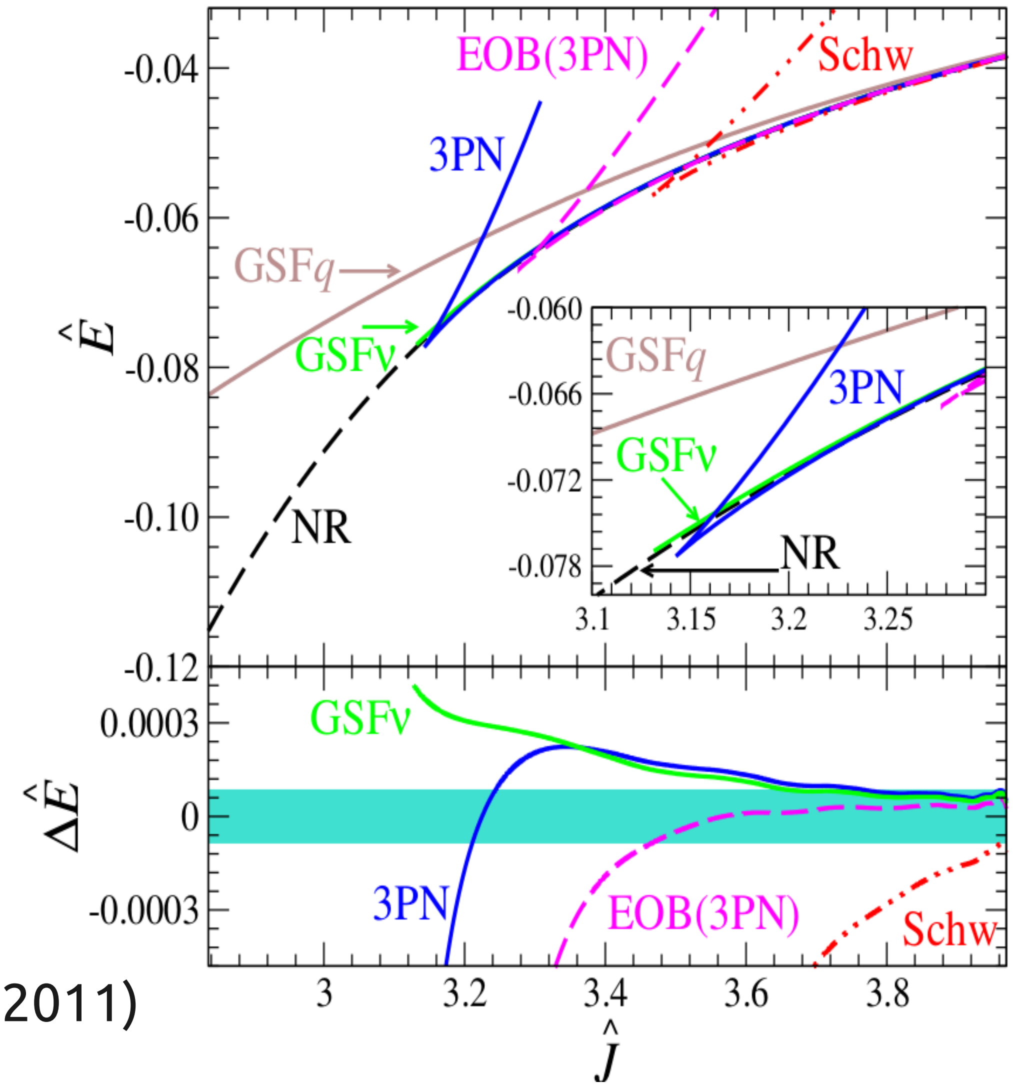
$$C_\Omega = \frac{1}{2} + \frac{1}{4\sqrt{2}}\left[\frac{1}{3} z''_{\rm SF}\left(\frac{1}{6}\right) - z'_{\rm SF}\left(\frac{1}{6}\right)\right] = 1.2510(2).$$

➔ Agreement at 1σ level!

➔ More accurate method (only requires to evaluate regular function for circular orbits, instead of stability analysis of eccentric orbits around unstable point)

Comparison to NR and PN theory

- NR: extract E(Ω) and J(Ω) from inspiral/plunge simulation (Damour et al 2012)
- 3PN, EOB(3PN) expressions for E(Ω) and J(Ω)
- GSFν: expressions for E(Ω) and J(Ω) we just derived
- GSFq: expressions just derived, but in terms of $q=m_1/m_2=\nu+O(\nu)^2$
- Schw= test-particle limit (no SF)
- $\nu\in(0,1/4)$ vs $q\in(0,1)$ explains GSFν performance: same observed for
  - SF effect on precession (Le Tiec et al 2011)
  - GW fluxes (Detweiler 1979)



Calculation on unknown terms in PN theory

- SF contribution to E and J contains in principle all PN orders

→ calculate 4PN, 5PN and 6PN terms (quadratic in ν) in the PN binding energy and angular momentum (Le Tiec, Blanchet and Detweiler 2012)

$$\hat{E}(x) = -\frac{x}{2}\left\{1 + \left(-\frac{3}{4} - \frac{\nu}{12}\right)x + \left(-\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}\right)x^2 + \left(-\frac{675}{64} + \left[\frac{34445}{576} - \frac{205}{96}\pi^2\right]\nu \right.\right.$$
$$- \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3\Big)x^3 + \left(-\frac{3969}{128} + \nu e_4(\nu) + \frac{448}{15}\nu \ln x\right)x^4 + \left(-\frac{45927}{512} + \nu e_5(\nu)\right.$$
$$\left.\left. + \left[-\frac{4988}{35} - \frac{1904}{15}\nu\right]\nu \ln x\right)x^5 + \left(-\frac{264627}{1024} + \nu e_6(\nu) + \nu e_6^{\ln}(\nu) \ln x\right)x^6 + o(x^6)\right\}$$

$$e_4(0) = +153.8803(1) \qquad e_5(0) = -55.13(3)$$

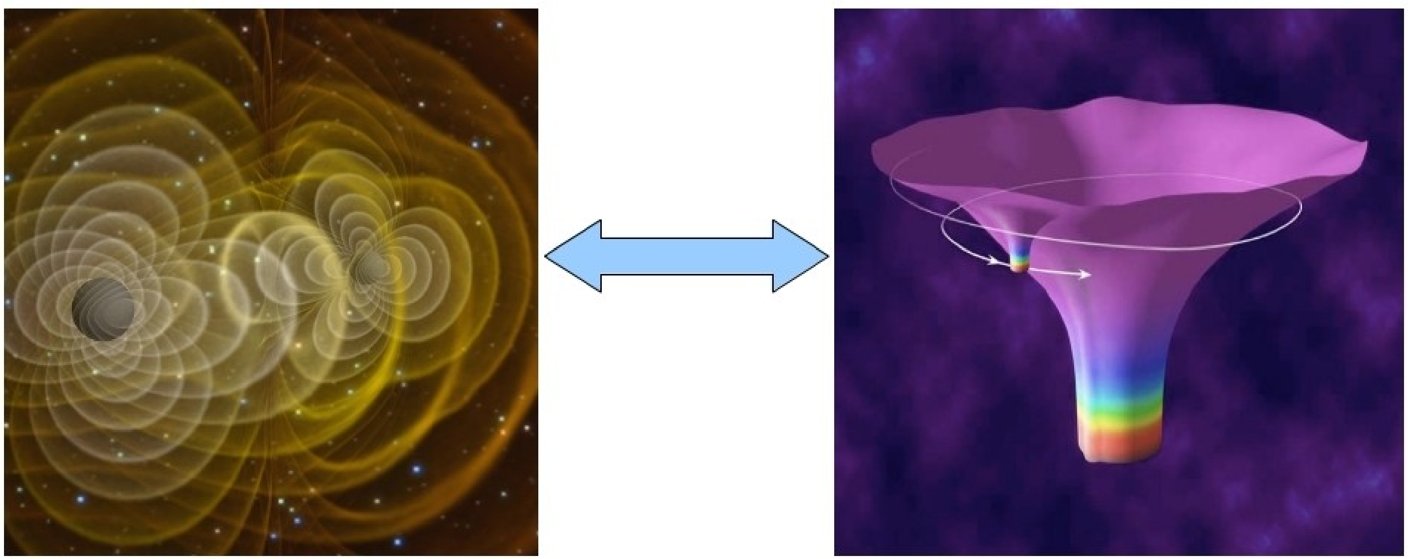
$$e_6(0) = +588(7) \qquad e_6^{\ln}(0) = -1144(2)$$

The effective-one-body formalism

- Motivations:

  - understand 2-body problem
  - fast and accurate templates for GW detectors

- Main idea: map 2-body problem into test-particle problem



$m_1 = m_2$ $\quad$ $m_1 \ll m_2$

Is this mapping possible?

- Newtonian non-spinning binaries can be mapped to non-spinning test-particle with mass $\mu = m_1 m_2/(m_1 + m_2)$ around mass $m = m_1 + m_2$

- Energy levels of positronium ($e^+ - e^-$) can be mapped to those of hydrogen through

$$\frac{E_{\rm H}}{\mu c^2} = \frac{E_{\rm pos}^2 - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4}$$

Mapping possible in PN theory for non-spinning BHs (Buonanno & Damour 2001)

- PN Hamiltonian in ADM coordinates → canonical transformation (does not affect physics) → "Real'' PN Hamiltonian $H_{PN,real}$
- Particle with mass $\mu= m_1 m_2/(m_1 +m_2)$ around a $m=m_1 +m_2$ *deformed* Schwarzschild BH ("effective problem'') has Hamiltonian H_{eff} , which satisfies
$$\frac{H_{\rm eff}}{\mu c^2} = \frac{H^2_{\rm PN,\,real} - m_1^2 c^4 - m_2^2 c^4}{2 m_1 m_2 c^4} \quad \text{(up to 3 PN)} \ (*)$$
- H_{eff} can be calculated at all PN orders (deformed Schwarzschild metric given at all PN orders) → invert Eq (*) and get "real" Hamiltonian valid at all PN orders:
$$H_{\rm real} = m\sqrt{1 + 2\frac{\mu}{m}\left(\frac{H_{\rm eff}}{\mu} - 1\right)}\,.$$

More on the EOB

- Mapping also possible for systems of *spinning* BHs in PN theory (Damour et al 2008, Barausse and Buonanno 2010, 2011, Nagar 2011)

- Can be supplemented with model for waveforms (and thus fluxes) and recipe to attach QNMs → inspiral-merger-ringdown waveforms for spinning BHs implemented in LIGO and used for searches (Taracchini et al 2012)

- EOB has free parameters: fixed by comparison to NR waveforms and SF results (PN theory already included in the mapping and waveforms)

The EOB conservative dynamics for non-spinning BHs

- Deformed Schwarzschild geometry given by

$$ds^2_{\rm eff} = -A(r)\,dt^2 + B(r)\,dr^2 + r^2 d\Omega^2$$

$$A(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu\, u^4 + \mathcal{O}(u^5)$$

$$\bar{D}(u) = 1 + 6\nu\, u^2 + (52 - 6\nu)\,\nu\, u^3 + \mathcal{O}(u^4) \qquad \bar{D} \equiv (AB)^{-1} \qquad u \equiv M/r$$

- Potentials can be rewritten in different ways at higher PN orders (e.g. Pade resummation, “log” resummation)
- Particle motion in this effective geometry described by

$$H^{\rm eff}(r, p_r, p_\phi) \equiv \mu\, \widehat{H}^{\rm eff} = \mu \sqrt{A(r)\left[1 + \frac{A(r)}{D(r)} p_r^2 + \frac{p_\phi^2}{r^2} + 2(4 - 3\nu)\,\nu\,\frac{p_r^4}{r^2}\right]}$$

(“non-geodesic” terms vanish at all orders for circular orbits)

- EOB dynamics given by “real” Hamiltonian

$$H_{\rm real} = M\sqrt{1 + 2\nu\left(\frac{H_{\rm eff}}{\mu} - 1\right)}$$

How to include the information from the SF in the EOB?

- Damour 2010, Barausse & Buonanno 2010: calibrate free parameters regulating high (unknown) PN orders in A(u) with SF results for ISCO shift
- Barack, Damour & Sago 2010: SF result for periastron precession constrains combination of A(u) and D(u)

Results for redshift observable as function of z unused!

Now: write EOB potentials as

$$A(u) = 1 - 2u + \nu\, A_{\rm SF}(u) + \mathcal{O}(\nu^2)\,,$$
$$\bar{D}(u) = 1 + \nu\, \bar{D}_{\rm SF}(u) + \mathcal{O}(\nu^2)\,.$$

- Compare EOB binding energy (at the "SF" order) with expression coming from redshift observable ⟶ $A_{SF}(u)$
- Combine with SF result for periastron precession ⟶ $D_{SF}(u)$

The EOB and SF binding energies

- SF: $E_{\rm SF}(x) = \frac{1}{2} z_{\rm SF}(x) - \frac{x}{3} z'_{\rm SF}(x) - 1$

$$+ \sqrt{1-3x} + \frac{x}{6} \frac{7-24x}{(1-3x)^{3/2}},$$

$$J_{\rm SF}(x) = -\frac{1}{3\sqrt{x}} z'_{\rm SF}(x) + \frac{1}{6\sqrt{x}} \frac{4-15x}{(1-3x)^{3/2}}. \qquad x \equiv (m\Omega)^{2/3}$$

- EOB: from $H_{\rm real} = M\sqrt{1+2\nu\left(\frac{H_{\rm eff}}{\mu}-1\right)}$ and Hamilton eqs, calculate energy for circular orbits as function of *x*

$$\hat{E}_{\rm EOB}(x) = \frac{1-2x}{\sqrt{1-3x}} - 1 + \nu\left\{\frac{1-4x}{(1-3x)^{3/2}}\frac{A_{\rm SF}(x)}{2} - \frac{x}{\sqrt{1-3x}}\frac{A'_{\rm SF}(x)}{3} - \left(\frac{1-2x}{\sqrt{1-3x}} - 1\right) \times \left[\frac{x}{3}\frac{1-6x}{(1-3x)^{3/2}} + \frac{1}{2}\left(\frac{1-2x}{\sqrt{1-3x}} - 1\right)\right]\right\} + \mathcal{O}(\nu^2)$$

Comparing the binding energies

$$\hat{E}_{SF}(x) = \hat{E}_{EOB}(x) \rightarrow$$

$$2x\, A'_{\rm SF}(x) - 3\frac{1-4x}{1-3x} A_{\rm SF}(x) = x\,\frac{1-6x}{1-3x} + \sqrt{1-3x}\times$$

$$\left[2x\, z'_{\rm SF}(x) - 3z_{\rm SF}(x) + x\,\frac{1-5x+12x^2}{(1-3x)^2}\right] \rightarrow$$

$$A_{\rm SF}(x) = \sqrt{1-3x}\; z_{\rm SF}(x) - x\left(1 + \frac{1-4x}{\sqrt{1-3x}}\right)$$

- Because $\frac{\partial E_{EOB}}{\partial J_{EOB}} = \Omega = \frac{\partial E_{SF}}{\partial J_{SF}}$, sufficient to ensure $\hat{J}_{EOB}(x) = \hat{J}_{SF}(x)$
- Automatically reproduces SF ISCO shift
- Completely determines A(u) and EOB dynamics for circular orbits at linear order in mass ratio!

The radial potential D

- SF effect on periastron precession

$$\Omega_r \equiv \frac{2\pi}{P} \qquad \Omega_\phi \equiv \frac{1}{P}\int_0^P \dot{\phi}(t)\,\mathrm{d}t = K\,\Omega_r \qquad \Delta\Phi/(2\pi) = K - 1$$

$$W \equiv 1/K^2 = 1 - 6x + \nu\,\rho_{\rm SF}(x) + \mathcal{O}(\nu^2)$$

$$\rho_{\rm SF}(x) = 4x\left(1 - \frac{1-2x}{\sqrt{1-3x}}\right) + A_{\rm SF}(x) + x\,A'_{\rm SF}(x) + \frac{x}{2}(1-2x)\,A''_{\rm SF}(x) + (1-6x)\,\bar{D}_{\rm SF}(x)\,. \quad (2$$

- $\rho_{SF}(x)$ known as fit
- Inversion gives $D_{SF}(x)$
- Completes knowledge of EOB metric at linear order in mass-ratio

Check converge of PN orders

- EOB potentials A(u) and D(u) now known at all PN orders (at linear order in mass-ratio)
- We can truncate them at nPN order and compare performance of EOB vs PN theory

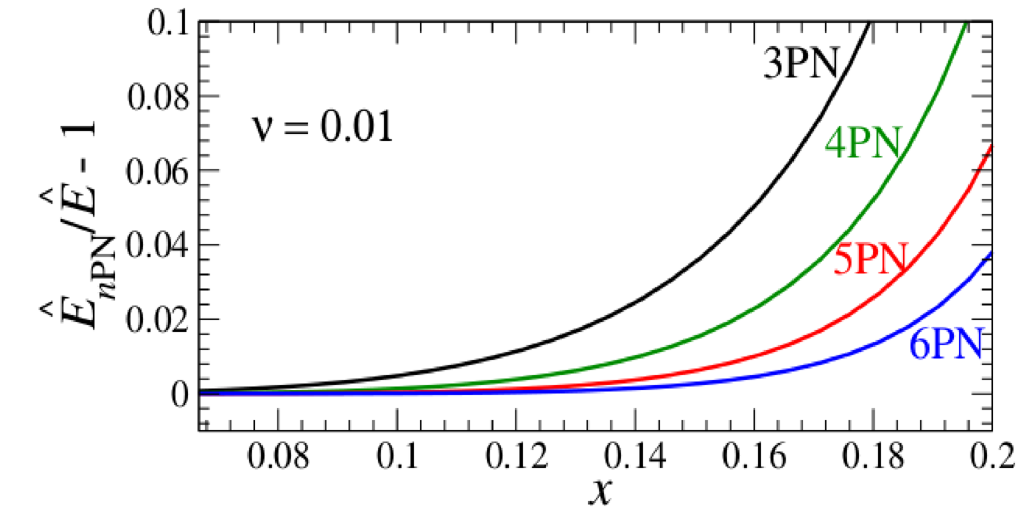
$$A(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41\pi^2}{32}\right)\nu\, u^4 + \nu\left[a_5(\nu) + a_5^{\ln}(\nu)\,\ln u\right]u^5 + \nu\left[a_6(\nu) + a_6^{\ln}(\nu)\,\ln u\right]u^6 + \nu\left[a_7(\nu) + a_7^{\ln}(\nu)\,\ln u\right]u^7 + o(u^7)$$

$$\bar{D}(u) = 1 + 6\nu\, u^2 + (52 - 6\nu)\,\nu\, u^3 + \nu\left[\bar{d}_4(\nu) + \bar{d}_4^{\ln}(\nu)\,\ln u\right]u^4 + \nu\left[\bar{d}_5(\nu) + \bar{d}_5^{\ln}(\nu)\,\ln u\right]u^5 + o(u^5)$$

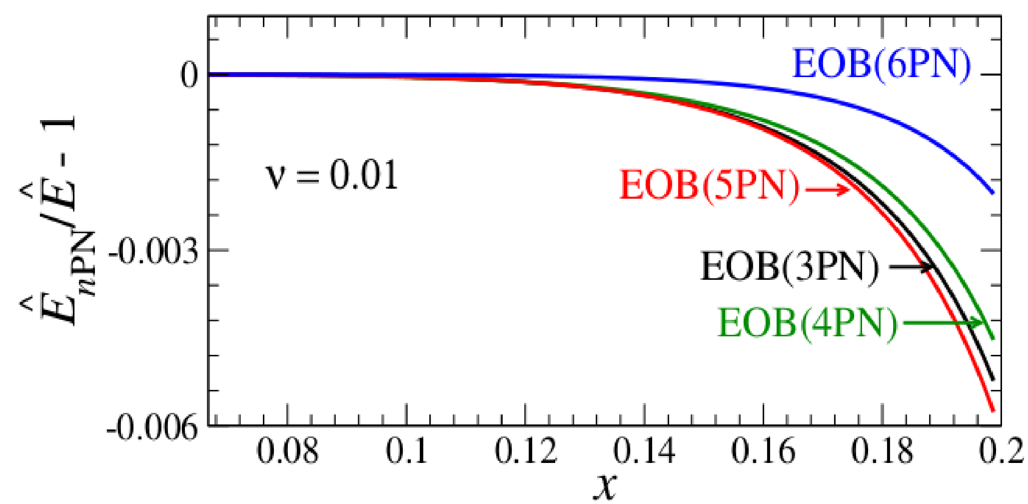
$$a_5(0) = +23.50190(5)\,,\quad a_6(0) = -131.72(1)\,,\quad a_7(0) = +118(2)\,,\quad a_7^{\ln}(0) = -255.0(5)\,.$$

$$\bar{d}_4(0) = +226.0^{+7}_{-4}\,,\quad \bar{d}_4^{\ln}(0) = +\frac{592}{15}\,,\quad \bar{d}_5(0) = -649^{-1200}_{+400}\,,\quad \bar{d}_5^{\ln}(0) = -\frac{1420}{7}\,.$$

EOB and PN vs approximants for the binding energy

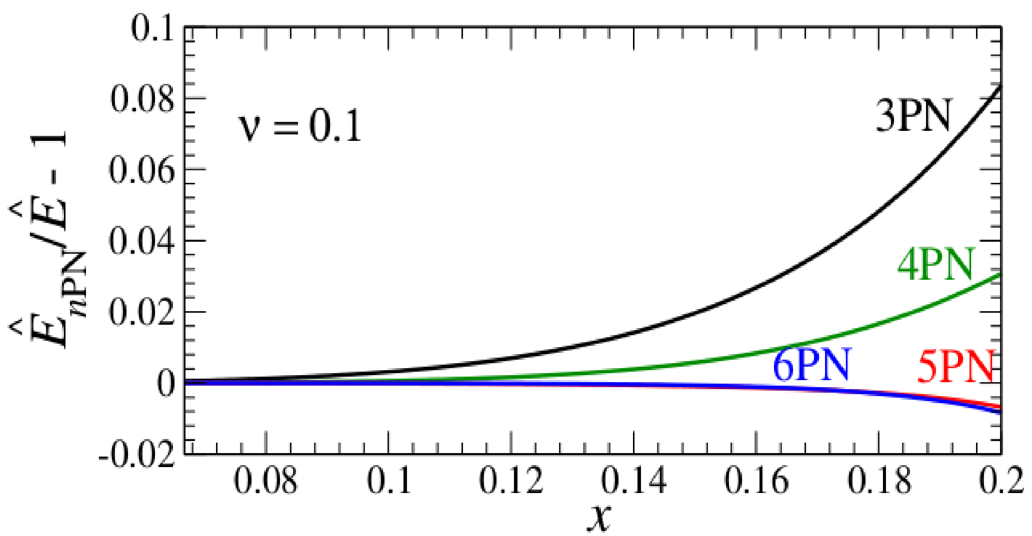


PN

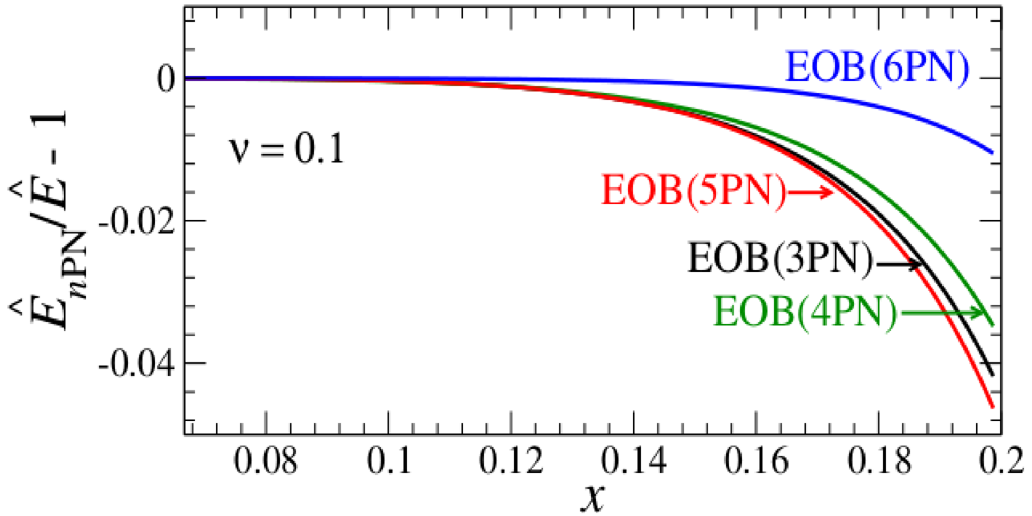


EOB

EOB and PN vs approximants for the binding energy

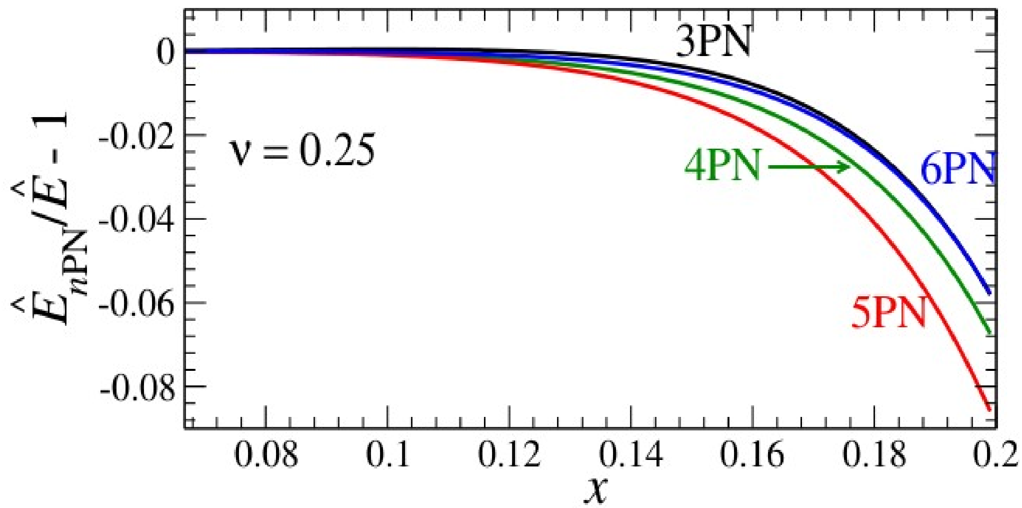


PN

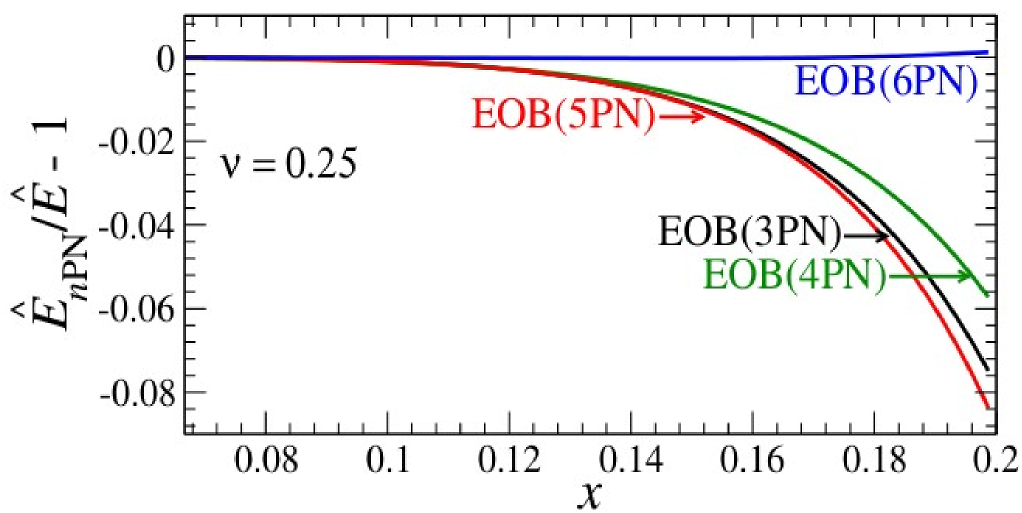


EOB

EOB and PN vs approximants for the binding energy



PN



EOB

Conclusions

- Calculated binding energy and angular momentum of circular binary of compact objects, at next-to-leading order in mass-ratio
- Recovered shift of ISCO of particle in Schwarzschild under effect of conservative self-force
- Binding energy, when expressed in terms of symmetric mass-ratio, works also for comparable-mass binaries
- Calculated EOB metric exactly at linear order in mass ratio

THE END