Ringdown amplitudes in extreme mass ratio inspiral

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Outline

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1. Motivation & Background
2. Non-rotating BHs
3. Comparing to numerical results

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Numerical simulation of equal mass BH merger (Caltech/Cornell/CITA)

Motivation

- Gravitational wave observatories $\Rightarrow$ Theoretical templates of possible GW signals from compact binary systems.
- Improve theoretical understanding of BH physics, GR 2-body problem.

Background

- Analytical control of 2-body problem in GR:
  - Post-Newtonian (PN) approximation $\Leftrightarrow v \ll c$.
  - Extreme mass ratio (EMR) limit $\Leftrightarrow m_1 \ll m_2$.
- Stages of EMR inspiral:
  - *Adiabatic inspiral* - the system moves on quasibound orbits and loses energy slowly to GW.
  - *Plunge* of the small compact object into the large BH, followed by "ringdown" of the final BH into its steady state.

Background

- The current state-of-the-art waveform templates ("Effective one body") model ringdown waveform as general superposition of the BH's quasinormal modes (QNMs). *We computed this late time signal from the theory.*
- As a compact object orbits a BH it radiates away its eccentricity, *circularizing* the orbit. $\Rightarrow$ Consider a special orbit - the *post-ISCO* plunge - starting from a circular orbit at the innermost stable circular orbit.

- The problem: compute the late time/ringdown signal emitted in a post-ISCO plunge, as a superposition of QNMs.

Black hole perturbation theory

(SH, Kol, PRD '11)

- Schwarzschild metric

$$ds^2 = g^{Schw}_{\mu\nu} dx^\mu dx^\nu = -f\, dt^2 + f^{-1}\, dr^2 + r^2\, d\Omega^2 \quad ; \quad f := 1 - \frac{r_s}{r}$$

- Small perturbation $g_{\mu\nu} = g^{Schw}_{\mu\nu} + h_{\mu\nu}$.
- Decompose into (tensor) spherical harmonics.

Black hole perturbation theory

- Plug into Einstein equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}$.
- Gauge-invariant masterfunctions $\Psi_{o/e}$ obey Regge-Wheeler (odd parity) and Zerilli (even parity) equations.

(odd-parity) Perturbation equations

$$\left(\Box - V^{\ell}_{o/e}\right)\Psi^{\ell m}_{o/e} = S^{\ell m}_{o/e}$$

$$\Box = \partial^2_{r_*} - \partial^2_t$$

$$dr_* := \frac{dr}{f}$$

Tortoise coordinate: $-\infty < r_* < \infty$.

$$\Psi_{o/e} = \Psi_{o/e}(h_{\mu\nu})$$

$$V^{\ell}_{o} := f\left[\frac{\ell(\ell+1)}{r^2} - \frac{3r_s}{r^3}\right]$$

Solving the wave equation

- Frequency domain - 1 D problem.

$$\left(\partial_{r_*}^2 + \omega^2 - V_{o/e}^{\ell\omega}\right) \Psi_{o/e}^{\ell m\omega} = S_{o/e}^{\ell m\omega}$$

- Boundary conditions - outgoing/retarded.

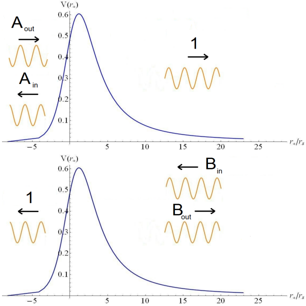
$$\Psi_{o/e}^{\ell m\omega} \sim \exp\left(i\omega r_*\right) \quad ; \quad r_* \to \infty$$
$$\Psi_{o/e}^{\ell m\omega} \sim \exp\left(-i\omega r_*\right) \quad ; \quad r_* \to -\infty$$

- Solution in terms of Green's function.

$$\Psi^{\ell m\omega}(r) = \frac{1}{2\pi} \int_{r_s}^{3r_s} dr'\, G^{\ell\omega}(r_*, r'_*)\, S^{\ell m\omega}(r'_*)$$

Solving the wave equation

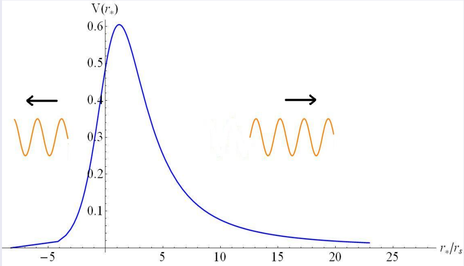
- Construct Green's function from homogeneous solutions.
  - u_∞ :
  - u_{hor} :



$$G^{\ell\omega}(r_*, r_*') = \frac{u_{hor}(r_*')u_\infty(r_*)}{2i\omega B_{in}(\omega)}$$

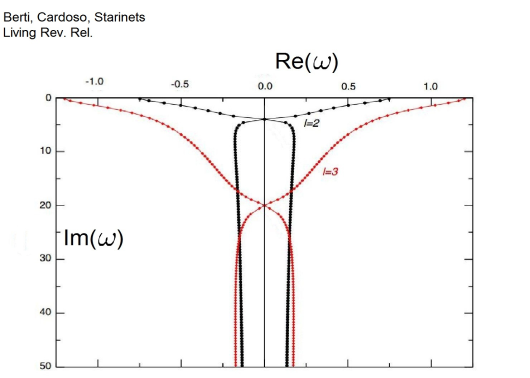
Quasinormal modes

- For a specific set of frequencies, $B_{in}(\omega_n) = 0 \ \Rightarrow$ poles of G.
- Homogeneous solutions with purely outgoing boundary conditions.
- Near QNMs $B_{in} \simeq \beta_{n\ell}\left(\omega - \omega_{n\ell}\right)$ $\qquad \beta_{n\ell} := \partial B_{in}|_{\omega_{n\ell}}$.



Quasinormal modes

QNM spectrum for $\ell = 2, 3$



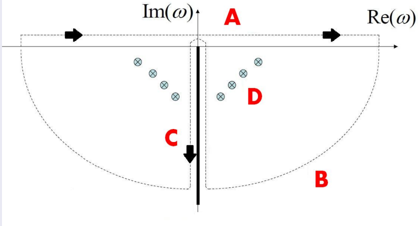
Quasinormal modes

- Back to the time domain, the solution is

$$\Psi^{\ell m}(r,t) = \frac{1}{2\pi}\int_{r_s}^{3r_s} dr' \left[\int_{-\infty}^{\infty} e^{-i\omega t}\, G^{\ell}(\omega, r_*, r'_*)\, S^{\ell m}(r'_*, \omega)\, d\omega\right]$$

$$= \sum_n \frac{1}{2\omega_{n\ell}\beta_{n\ell}} e^{-i\omega_{n\ell}(t-r_*)} \int_{r_s}^{3r_s} dr'\, u_{hor}(r'_*)\, S^{\ell m}(r'_*, \omega_{n\ell})$$

- (A) - Real frequencies
- (B) - Prompt emission
- (C) - Power-law tail
- (D) - QNMs



Ringdown amplitudes

$$\Psi^{\ell m}(r,t) = \sum_n C_{n\ell m} e^{-i\omega_{n\ell}(t-r_*)}$$

$$C_{n\ell m} := \frac{1}{2\,\omega_{n\ell}\,\beta_{n\ell}} \int_{r_s}^{3r_s} u^{n\ell}(r_*)\, S^{\ell m}(r_*,\omega_{n\ell})\, dr$$

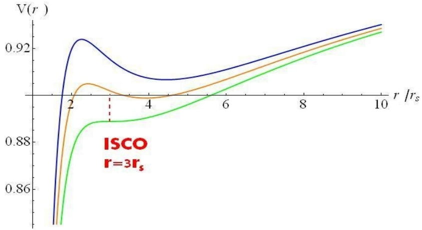
- Solve numerically for $\omega_{n\ell}$, $\beta_{n\ell}$, $u^{n\ell}(r_*)$.
- Solve analytically for $S^{\ell m}(r_*,\omega_{n\ell})$ - find the trajectory.

- BH is an open resonant cavity for gravitational perturbations.
- Late time signal composed of its QNMs with amplitudes given by their overlap with source.
- Intuition in the eikonal limit ($\ell \gg 1$) - light rays emitted from falling particle "trapped" in vicinity of light ring until released to infinity.

Numerical simulation of equal mass BH merger (Pretorius '05)

Post-ISCO Trajectory

- ISCO is an attractor - orbits tend to circularize.



- Marginally (un)stable orbit ⇒ ISCO parameters: $\tilde{E}_{\rm ISCO} \equiv \frac{2\sqrt{2}}{3}$, $\tilde{L}_{\rm ISCO} \equiv \sqrt{3}\, r_s$.

- Use symmetry - Killing vectors - to write geodesic equations

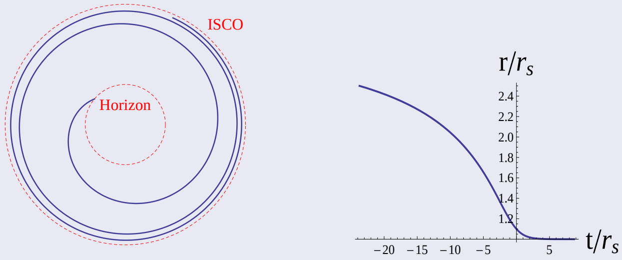
$$\begin{aligned} -\tilde{E} &:= g_{t\mu}\dot{x}^\mu \\ \tilde{L} &:= g_{\phi\mu}\dot{x}^\mu \\ -1 &= g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu \end{aligned}$$

Post-ISCO Trajectory

- Exact analytic solution

$$\frac{3r_s}{r} = 1 + \frac{12}{(\phi - \phi_0)^2} \qquad \chi := \frac{1}{2}\left(\frac{r_{ISCO}}{r} - 1\right)$$

$$t(r)/r_s = \frac{2\,r\left(1 - 12\,\frac{r_s}{r}\right)}{r_s\sqrt{\chi}} - 22\sqrt{2}\tan^{-1}\left(\sqrt{2\chi}\right) + 2\tanh^{-1}\left(\sqrt{\chi}\right)$$



Source term

- The source term is what couples to the gauge invariant masterfunction in the action

$$\sum_{\ell m} \int \psi^{\ell m} S^{\ell m}$$

$$S^{\ell m} = S^{\ell m}(T_{\mu\nu})$$

- Determined from stress-energy of infalling compact object. We use its general form (Martel & Poisson '05) to compute for our plunging compact object. Full analytical computation performed.

Numerical evaluation

- $\omega_{n\ell}$, u^{ℓ}, $\beta_{n\ell}$ were numerically obtained using Leaver's continued fraction method: factor out analytic behavior at infinity and expand

$$\psi_{e/o}^{\ell m} = e^{i\omega(r-2)} r^{2i\omega} (r-1)^{-i\omega} \sum_k a_k \left(\frac{r-1}{r}\right)^k$$

- Plug into RW/Z equations. Obtain recurrence relations

$$\begin{cases} 0 = \alpha_0 a_1 + \beta_0 a_0 \\ 0 = \alpha_k a_{k+1} + \beta_k a_k + \gamma_k a_{k-1} \qquad k = 1, 2, \ldots \end{cases}$$

- α_k , β_k , γ_k are functions of (ℓ, m, ω, k).

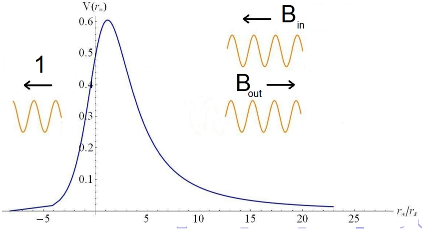
Numerical evaluation

- Boundary conditions require that the solution a_n be minimal and that the QNM frequency ω_n is the nth root of the continued fraction equation

$$0 = \beta_0 - \frac{\alpha_0\gamma_1}{\beta_1 - \frac{\alpha_1\gamma_2}{\beta_2 - \frac{\alpha_2\gamma_3}{\beta_3 - \dots}}}$$

- $\omega_n \Rightarrow$ generate a_k's using recursion relations $\Rightarrow$ construct $\psi^{n\ell m}_{e/o}$.

- Calculation of $\beta_{n\ell}$:
  - Expand wavefunction at infinity using Coulomb wavefunctions
  - Match at intermediate r to get $B_{in}(\omega_n + \delta\omega)$
  - Find derivative



Regularization

$$C_{n\ell m} := \frac{1}{2\,\omega_{n\ell}\,\beta_{n\ell}} \int_{r_s}^{3r_s} u^{n\ell}(r_*)\, S^{\ell m}(r_*, \omega_{n\ell})\, dr$$

Integral diverges!

$$(Integrand) \propto (r - r_s)^{-2i\frac{\omega}{r_s}} \quad ; \quad r \to r_{\rm hor}$$

$\Rightarrow$ Diverges for $\Im\left(\frac{\omega}{r_s}\right) \leq -\frac{1}{2}$.

Regularization

$$C_{n\ell m}\big|_{regularized} := \frac{1}{2\,\omega_{n\ell}\,\beta_{n\ell}} \int_{r_s+\epsilon}^{3r_s} \left(u^{n\ell}(r_*)\,S^{\ell m}(r_*,\omega_{n\ell}) - f\right)\,dr + F\big|_{r=3r_s}$$

- Essentially analytic continuation: subtract parts $\propto \epsilon^{-\#}$.

$$C_{n\ell m} = C_{n\ell m}\big|_{regularized} + \mathcal{O}\left(\epsilon^{-\#}\right)$$

- Choose f that would be easily integrable - in order to find $F\big|_{r=3r_s}$.

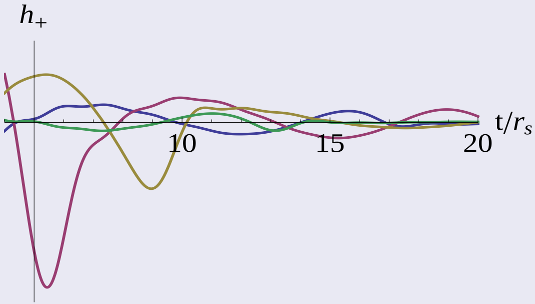
$$f = \left(\frac{r}{r_s} - 1\right)^{-2i\frac{\omega}{r_s}} \left(A_0 + A_1(r - r_s) + ...\right)$$

- Integrate numerically.

Results

- Complex numerical values obtained for leading $C_{n\ell m}$.
- Waveform at infinity is $h_{AB} = r \sum_{\ell m} \left(\Psi_{\mathrm{e}}^{\ell m} Y_{AB}^{\ell m} + \Psi_{\mathrm{o}}^{\ell m} X_{AB}^{\ell m} \right)$.

Results: Late time waveform



Results: Amplitudes

$l = 2$

m	$n = 1$	$n = 2$
2	$-0.0985724 - 0.747787i$	$-0.229354 + 0.428849i$
1	$-0.0210521 + 0.399297i$	$0.441304 - 0.31877i$
0	$-0.0887841 + 0.0979244i$	$0.303357 + 0.0416042i$
−1	$0.0274099 + 0.00889306i$	$-0.0324417 - 0.0903849i$
−2	$(-0.735088 + 3.59504i) \times 10^{-3}$	$0.0135078 - 0.0089118i$

$l = 3$

m	$n = 1$	$n = 2$
3	$-0.0375476 + 0.141024i$	$0.117107 - 0.057957i$
2	$0.0336116 - 0.0585892i$	$-0.104903 + 0.00738472i$
1	$0.0186212 - 0.0145793i$	$-0.0499562 - 0.0209467i$
0	$(-7.36412 + 0.205584i) \times 10^{-3}$	$0.0119392 + 0.0185584i$
−1	$(-1.32958 - 1.74202i) \times 10^{-3}$	$-2.4849 + 7.94279i) \times 10^{-3}$
−2	$(-3.59688 + 3.99339i) \times 10^{-4}$	$(2.43071 + 0.097078i) \times 10^{-3}$
−3	$(-5.2293 - 6.09408i) \times 10^{-5}$	$(-0.284844 + 4.19003i) \times 10^{-4}$

$l = 4$

m	$n = 1$	$n = 2$
4	$0.0318095 - 0.0308272i$	$-0.0489266 - 0.00923324i$
3	$-0.0165716 + 0.00919402i$	$0.0289206 + 0.0167393i$
2	$(-6.86211 + 1.83095i) \times 10^{-3}$	$0.0114492 + 0.0128208i$
1	$(2.32275 + 0.593617i) \times 10^{-3}$	$(-1.76906 - 7.17998i) \times 10^{-3}$
0	$(4.47817 + 5.89624i) \times 10^{-4}$	$(1.10773 - 2.61122i) \times 10^{-3}$
−1	$(0.514994 - 2.12774i) \times 10^{-4}$	$(-9.22639 + 3.19073i) \times 10^{-4}$
−2	$(6.24213 - 1.42382i) \times 10^{-5}$	$(-2.49301 - 2.12194i) \times 10^{-4}$
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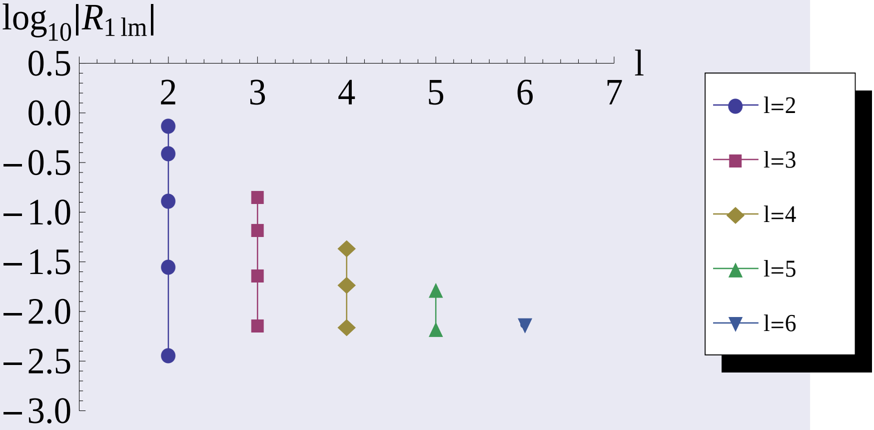
Results: Magnitude of excited QNMs



Comparing with numerical calculations

(Berti, Cardoso, SH, Kol, PRD '11)

- Obtained plunge signal by directly integrating EOM of projectile from $(r_{ISCO} - \epsilon)$ & gravitational perturbations for each (ℓ, m).
- Solved field equations in the frequency and transformed back to the time domain

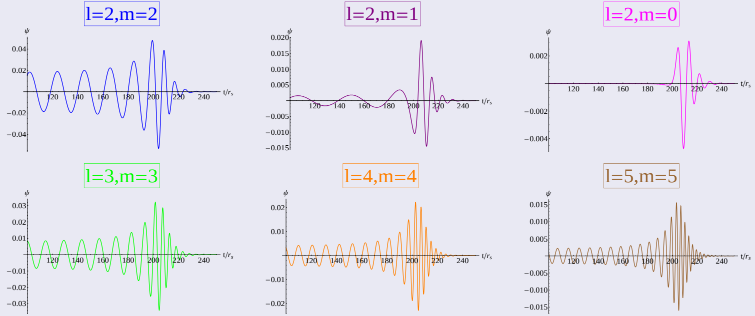
$$(\partial_{r_*}^2 + \omega^2 - V_{o/e}^{\ell\omega})\, \Psi_{o/e}^{\ell m\omega} = S_{o/e}^{\ell m\omega}$$

$$\Psi^{\ell m}(r,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-r_*)} \left[\int_{r_s}^{3r_s} dr'\, G^{\ell}(\omega, r_*, r'_*)\, S^{\ell m}(r'_*, \omega) \right] d\omega$$

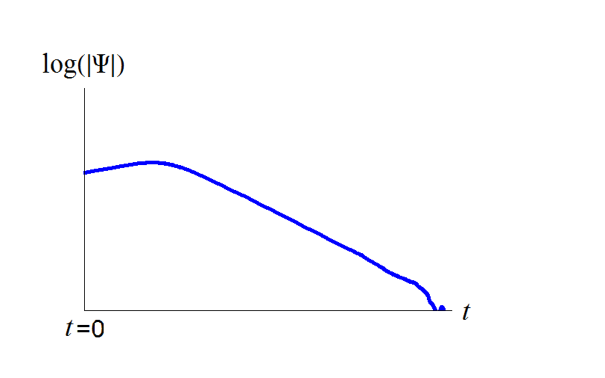
Comparing with numerical calculations

(Berti, Cardoso, SH, Kol, PRD '11)

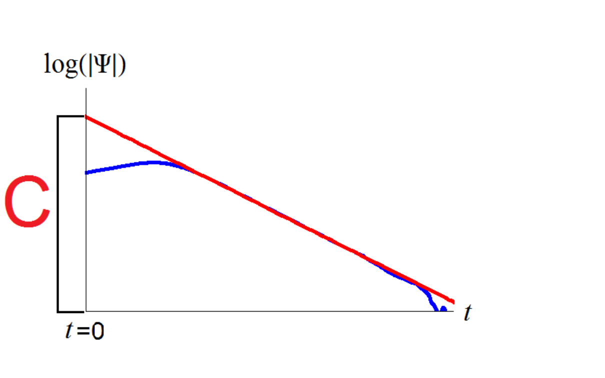
Numerical waveforms



Extracting amplitudes

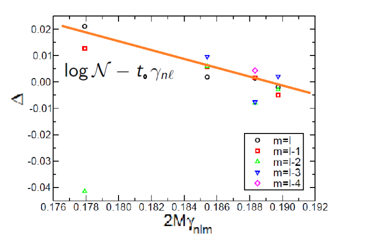


Extracting amplitudes



Comparing with numerical calculations

- Extracted amplitudes for different overtones one by one.
- Waveform composed of QNMs: $\psi_{\ell m} = \mathcal{N} \sum_n R_{n\ell m} \exp\left(i\omega_{n\ell m}(t - t_0)\right)$
- $$\Delta := \log\left(\left|C^{anlyt}/C^{num}\right|\right) \quad \leftrightarrow \quad \log\mathcal{N} - t_0\gamma_{n\ell}$$
  should be a linear function of $\gamma_{n\ell} := \Im\omega_{n\ell}$. Can find from it $\mathcal{N}$ (offset), t_0 (slope).
- Amplitudes agree to $< 1\%$!



Ringdown amplitudes in Kerr

(SH, Hewlett, Porfyriadis, Strominger, in progress)

- Generalize to rotating (Kerr) BHs. Clear motivation:
  - BHs in the sky are rotating, some quite rapidly.
  - Theoretical understanding of BHs - near extremal ones are simple. Kerr/CFT.
- Fields in Kerr background obey Teukolsky equations - enjoy (nontrivial) separability in 4D uncharged case

$$\begin{aligned} \psi &= \psi\left(h_{\mu\nu}\right) \\ &= \sum_{\ell m} \int d\omega e^{-i\omega\hat{t}+im\hat{\phi}} Y_\ell(\theta) u(\hat{r}) \end{aligned}$$

$$\begin{aligned} D_\theta^2 Y + \left(K_\ell - V^S\right) Y &= 0 \\ D_r^2 u + \left(V^{(R)} - K_\ell\right) u &= S \end{aligned}$$

Ringdown amplitudes in Kerr

- Solutions similar to Schwarzschild. Late time waveform is

$$\psi_{\ell m} = \sum_n C_n \, \hat{r}^3 e^{-i\omega_n(\hat{t}-r^*)+im\hat{\phi}} \, Y(\theta)$$

$$C_{n\ell m} := \frac{\pi}{\omega_{n\ell m}\beta_{n\ell m}} \int_{r_+}^{\infty} d\hat{r}' \, u_{n\ell m} \, S_{n\ell m}$$

nearNHEK geometry

- A Kerr BH is extremal when $a := J/M = M$.
- Near extremal: $\sqrt{M-a} \sim \tau_H := \frac{r_+ - r_-}{r_+} \ll 1$.
- Near extremality, transform

$$t = \lambda \frac{\hat{t}}{2M} \qquad r = \frac{\hat{r} - r_+}{\lambda r_+} \qquad \phi = \hat{\phi} - \frac{\hat{t}}{2M}$$

- Zoom in on near horizon region by $\lambda \to 0$ with $\alpha := \frac{\tau_H}{\lambda}$ fixed.
- Obtain nearNHEK metric

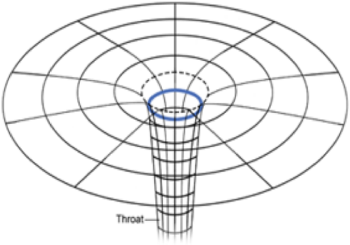
$$ds^2 = 2M^2\Gamma\left(-r(r+2\alpha)dt^2 + \frac{dr^2}{r(r+2\alpha)} + d\theta^2 + \Lambda^2\left(d\phi + (r+\alpha)dt\right)^2\right)$$

$$\Gamma(\theta) = \frac{1+\cos^2\theta}{2}, \quad \Lambda(\theta) = \frac{2\sin\theta}{1+\cos^2\theta}$$

nearNHEK geometry

- $r \gg \alpha$ limit $\Rightarrow$ NHEK geometry (exactly extremal, then zoom in).
- (near)NHEK was shown to be dual (entropy, scattering amplitudes) to a $(1+1)D$ thermal CFT. Resides at infinity of nearNHEK, where the throat is glued (field theory is coupled) to the far region.

- Excitation of QNMs takes place in the throat $\Rightarrow$ simplification.



Trajectory

- ISCO is at infinity of nearNHEK, but still in near horizon region.
- nearNHEK orbital parameters:

$$\tilde{e} = \frac{1}{\lambda}\left(2M\tilde{E} - \tilde{L}\right) = 0$$
$$\tilde{l} = \tilde{L} = \frac{2M}{\sqrt{3}}$$

- Trajectory

$$t(r) = -\frac{1}{2\alpha}\log r(r+2\alpha) + t_0$$
$$\phi(r) = \frac{3}{4\alpha}r + \frac{1}{2}\log\frac{r}{r+2\alpha} + \phi_0$$

Solving the radial equation

- Main idea - matched asymptotic expansion(MAE):
  - Near region $\frac{\hat{r}-r_+}{r_+} =: x \ll 1$.
  - Far region $x \gg \tau_H$.
- Solve separately for near and far regions, then match for $\tau_H \ll x \ll 1$: large x limit of u^{near} = small x limit of u^{far}.
- Take large $\hat{r}$ limit to find B_{in}, B_{out}.

nearNHEK QNMs

- Occur when $B_{in} = 0$ Solution for "long living" QNMs (Hod)

$$\omega_{QNM} = m\Omega_H + \frac{\tau_H}{4M}\left(m + i\chi - i(1/2 + n)\right)$$

- $\chi^2 := \frac{1}{4} + K_\ell - 2m^2$. χ can be imaginary ($m = \ell$ and large enough m) or real ($m < \ell$).
- New analytical results (general spin field) for QNM wavefunction u_n, excitation factor β_n, source term S_n.
- Excitation integral - Laplace transform of rational function - analytically calculable including regularization!

nearNHEK ringdown amplitudes

$$C_{n\ell m} = (\tau_H)^{\frac{1}{2}+\chi} \, \frac{\mu}{M^2} \, \frac{\left(\frac{-3im}{2}\right)^n}{n!} \, f(\ell, m)$$

- n dependence is very simple $\Rightarrow$ sum over n!

Ringdown waveform

$$\psi_{-2} = \hat{r}^3 Y^\ell e^{im\hat{\phi}} \sum_n e^{-i\omega_{n\ell m}(\hat{t}-r^*)} C_{n\ell m}$$

$$\simeq c_{\ell m}\ \hat{r}^3 Y^\ell e^{-i\omega_{\ell m}(\hat{t}-r^*)+im\hat{\phi}}$$

- Radial dependence eliminated!
- Calculate $f(\ell, m)$ in CFT.

$$c_{\ell m} := (\tau_H)^{\frac{1}{2}+\chi}\ \frac{\mu}{M^2}\ f(\ell, m)$$

$$\omega_{\ell m} := \left(\frac{m}{2M} - \tau_H \frac{3m}{8M}\right) - i\frac{\tau_H}{4M}(1/2+\chi)$$

Computing in NHEK

- NHEK is diffeomorphic to nearNHEK. The coordinate transformation is singular at the boundaries (implying they are not equivalent physically)

$$\begin{aligned} T &= -e^{-\alpha t}\frac{r+\alpha}{\sqrt{r(r+2\alpha)}} \\ R &= \frac{1}{\alpha} - e^{\alpha t}\sqrt{r(r+2\alpha)} \\ \Phi &= \phi - \frac{1}{2}\log\frac{r}{r+2\alpha} \end{aligned}$$

- This transformation takes the nearNHEK metric to NHEK

$$ds^2 = 2J\Gamma(\theta)\left[-R^2 dT^2 + \frac{dR^2}{R^2} + d\theta^2 + \Lambda(\theta)^2(d\Phi + R dT)^2\right]$$

and the nearNHEK plunge trajectory to

$$\begin{aligned} R &= R_0 \\ \Phi(T) &= -\frac{3}{4}R_0 T + \Phi_0 \end{aligned}$$

a circular orbit!

Computing in NHEK

- Solve in NHEK without specifying boundary conditions. General solution is

$$\psi = \sum_{\ell,m} e^{i(m\Phi+(3/4)R_0 T)} Y_\ell(\theta)\tilde{u}_{\ell m}(R)$$

- Solve for $\tilde{u}_{\ell m}$ with general BC - obtain $\tilde{u}_{\ell m}(R, c_1, c_2)$: c_1, c_2 are constants which depend on BC.
- Transform the result back to nearNHEK. Obtain *time domain* waveform there.
- Continue as before - match to far region of Kerr (MAE), get solution at infinity, Fourier transform to get QNM form.

Conclusions & Outlook

- Problem of late time observable radiation solved analytically & compared to numerical computation.
- Radiation is composed of BH eigenmodes, excited by plunging particle.
- In near extremal case - full analytical treatment possible. Astrophysically & holographically relevant!

Conclusions & Outlook

- Extensions & generalizations: different orbits, general BH spin, modified gravity, and more.
- Study effective field theories of NHEKs & more spacetimes. Classically integrate out bulk DOF (QNMs can help). What can we learn?
- Use QNMs for self-force computations. Simplify computation of late time tail contribution.