Numerical Relativity in higher dimensional spacetimes

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Outline of the talk

- Introduction
- Numerical Relativity in higher dimensional spacetimes
  - Decomposition of the Einstein equation
  - BSSN formalism
  - Gauge equations
- Black hole collision
- Black hole in AdS spacetimes
- Summary

Introduction

Veryfication for higher dimensional gravity

- In principle, BH can be formed when the energy is confined to the sufficient small region.(Cf. "Hoop-Conjecture" Thorne(1972))
- Planck scale is $10^{19}[\mathrm{GeV}]$ if $D = 4$.
- Higher dimensional theory, Arkani-Hamed+(98),Randall+(99)
- Current experiments show the inverse square law of the gravity is valid up to $0.1 \sim 0.01\mathrm{mm}$.
- Planck scale could be larger than 10 $[\mathrm{TeV}]$.
- Possibility of the BH formation for particle collisions

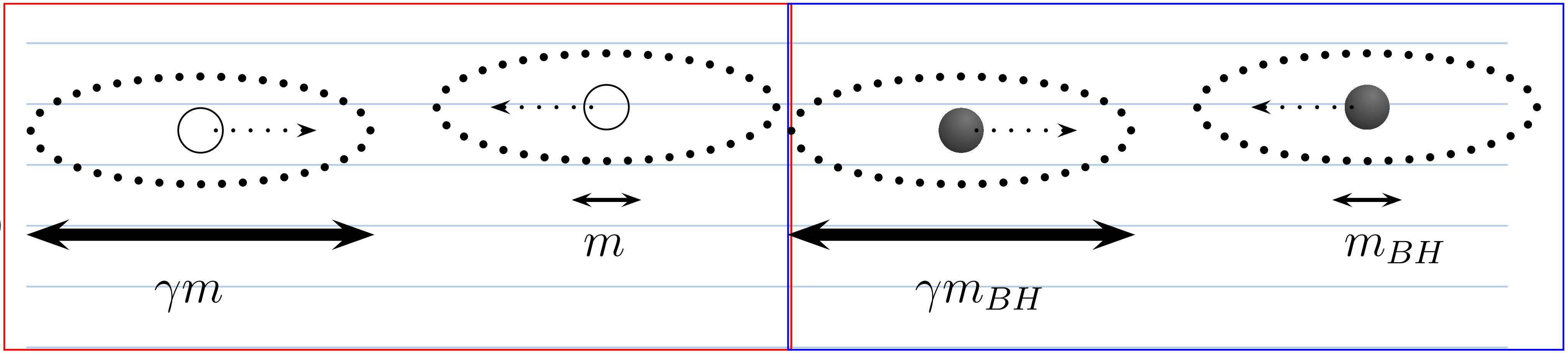
  Dimopoulos and Landsberg(2001), Giddings and Thomas(2002)

High-energy collision | BH formation? | evapolation

High-velocity Collisions in Higher Dimensions

It is believed that Gravity is dominant below the Planck scale.
$\longrightarrow$ We could treat it as classical gravity.

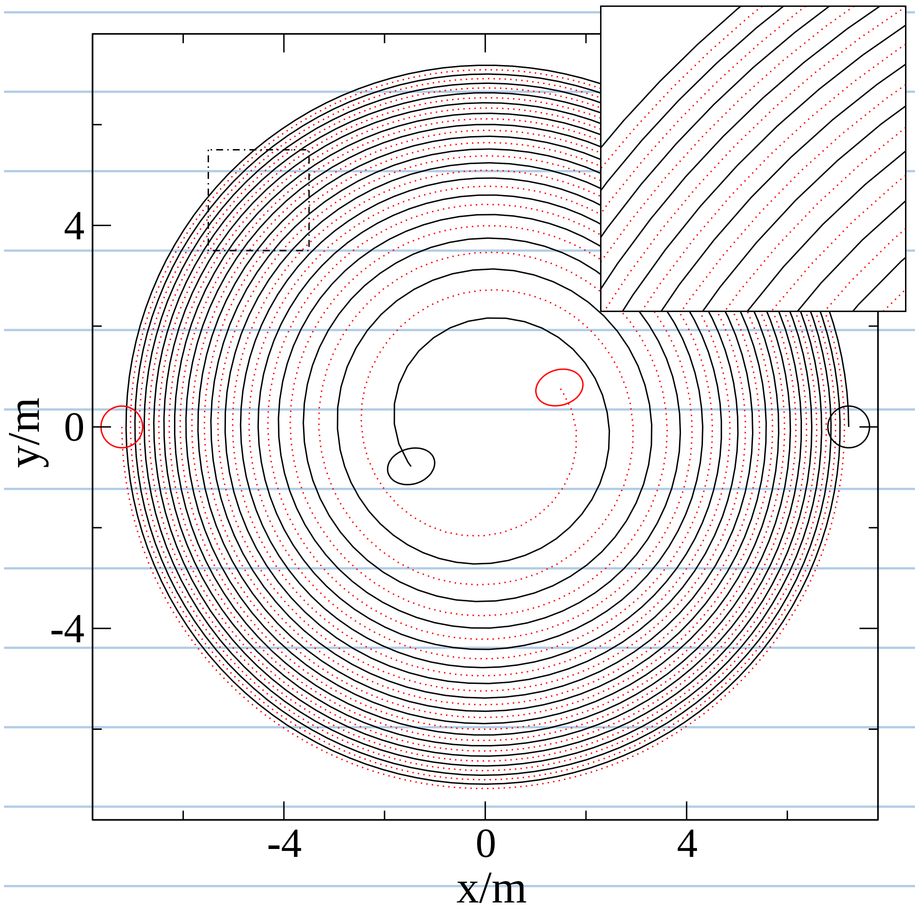
- Test particle collisions
- Shock wave collisions Penrose(1974),Eardley and Giddings(2002)
- High-velocity BH collisions Shibata+(2008),Sperhake+(2009)

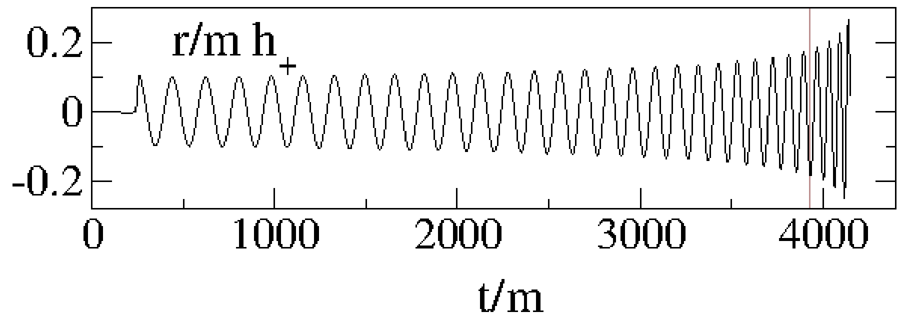


Impact paramter(BH formation)
M and J(feature of BH)
Dissipation(feature of GWs)

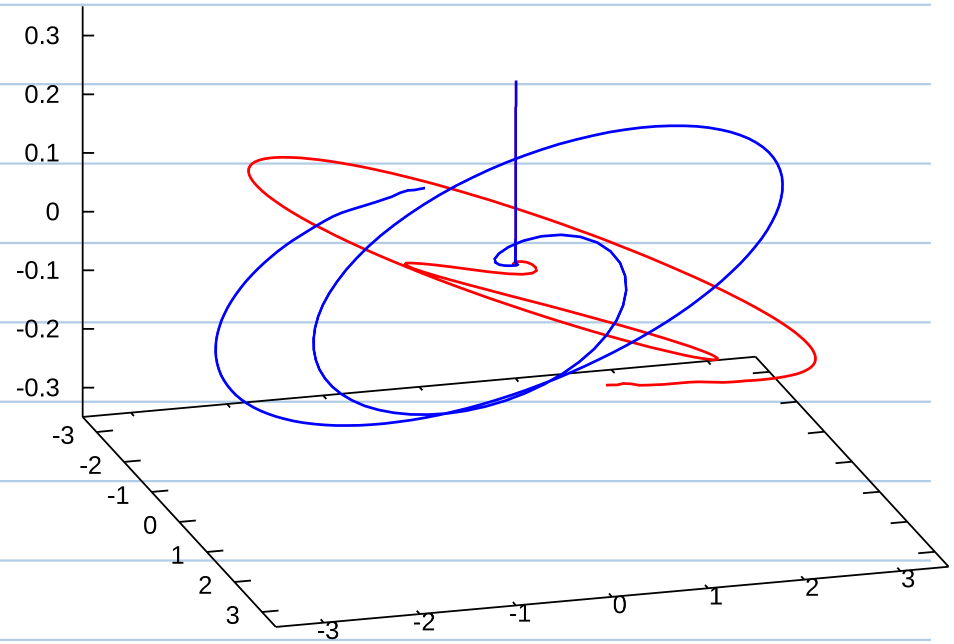
NR in Higher Dimensions

4D Numerical Relativity(Binary BHs)





Boyle,Brown,Kidder,Mroue,Pfeiffer,Scheel,Cook,Teukolsky(2007)

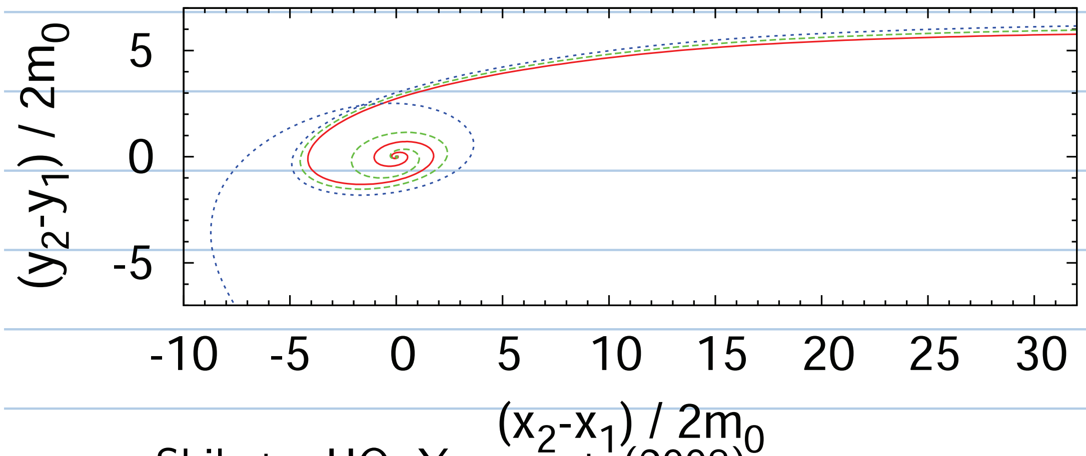


Campanelli,Lousto,Zlochower,Merritt(2007)

- The evolution of Binary BHs is calculated quite accurately.
- It's also popular to study about the Gravitational Recoil (Kick).

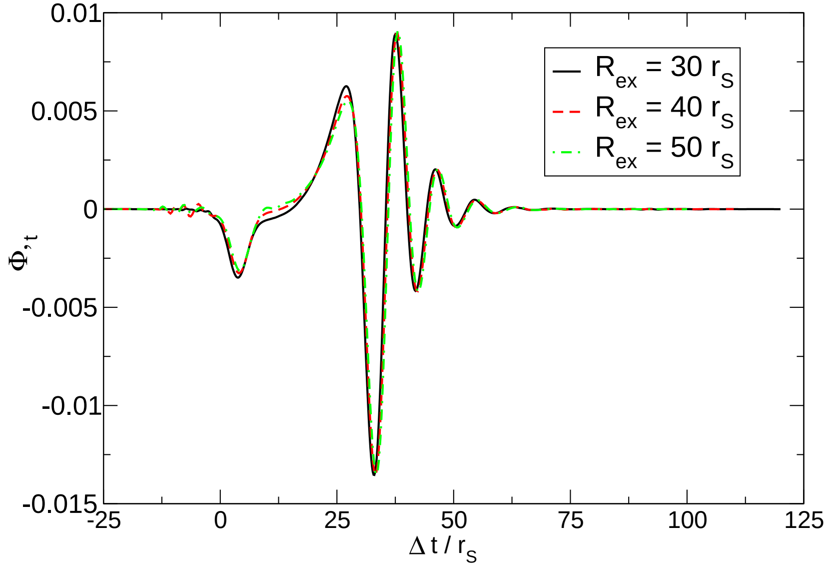
High-velocity BH Collisions in Numerical Relativity

Orbits of a BH in 4D collision

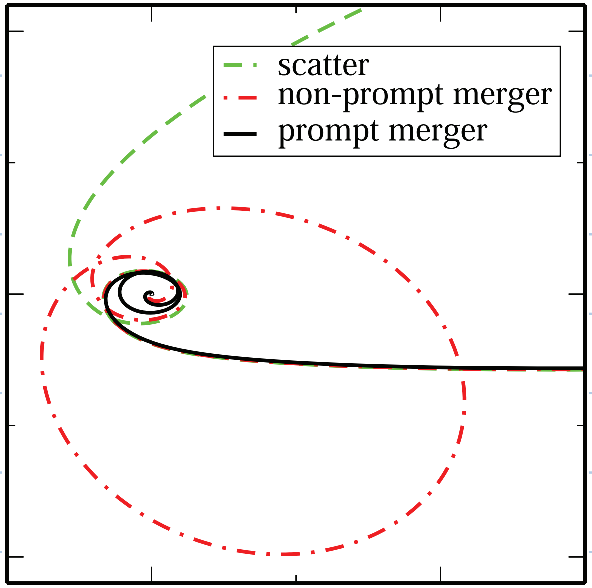


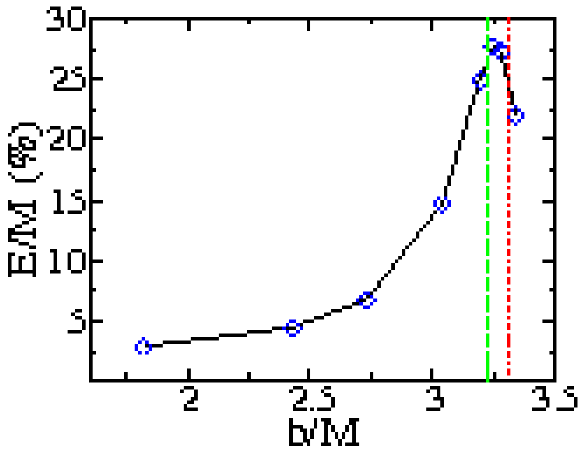
Shibata, HO, Yamamoto(2008)

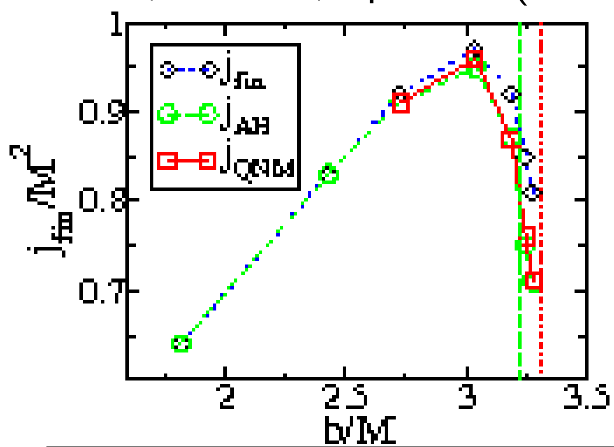
Waveform of 5D Headon collision



Witek, Zilhão, Gualtieri,Cardoso, Herdeiro, Nerozzi, Sperhake(2010)







Sperhake,Cardoso,Pretorius,Berti,Hinderer,Yunes(2009)

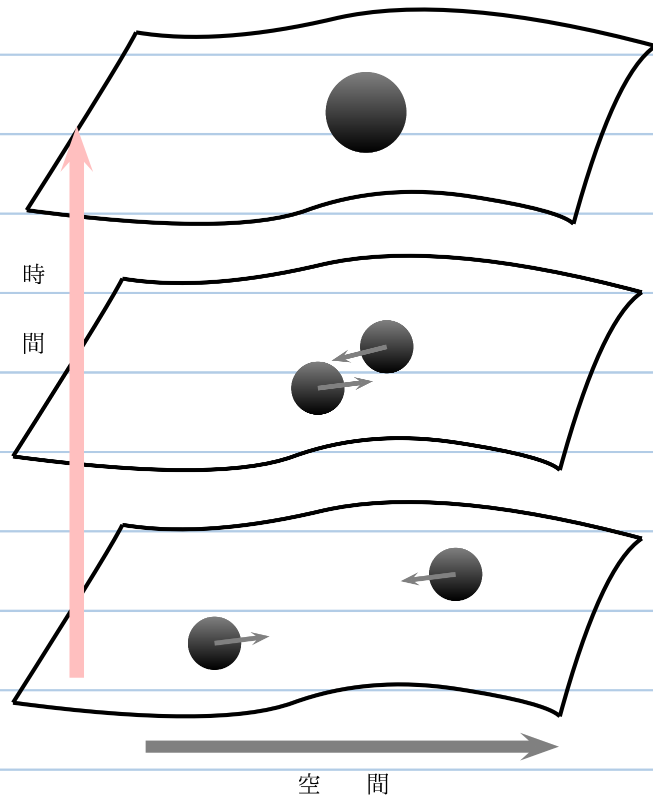
Spin of the formed BH in 4D collision

Numerical Relativity

D-dimensional NR(D-1+1 decomposision)

To solve the Einstein equation

$$\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = \kappa T_{ab} (= 0).$$



Spatial metric

$$\gamma_{ab} \equiv g_{ab} + n_a n_b$$

n^a:Normal vector to the space ($n^a = (1/\alpha, \beta^i/\alpha)$)

Extrinsic curvature

$$K_{ab} \equiv -\frac{1}{2\alpha} \left(\partial_t \gamma_{ab} - D_b \beta_a - D_a \beta_b \right)$$

Decomposition of the Einstein equation
$\longrightarrow$ **Same as 4D**

Evolution equation

$$\partial_t K_{ab} = -D_b D_a \alpha + \alpha \left(R_{ab} - 2K_{ac} K^c_b + K_{ab} K \right)$$
$$+\beta^c D_c K_{ab} + K_{cb} D_a \beta^c + K_{ca} D_b \beta^c$$

Constraints

Hamiltonian constraint
$R + K^2 - K_{ab} K^{ab} = 0$
Momentum constraints
$D_b K^b_a - D_a K = 0$

D-dimensional NR(BSSN formalism)

<u>BSSN formalism in D-dim.</u> Yoshino and Shibata (2009)

We cannot evolve γ_{ij} and K_{ij} numerically because of the existence of violating modes.

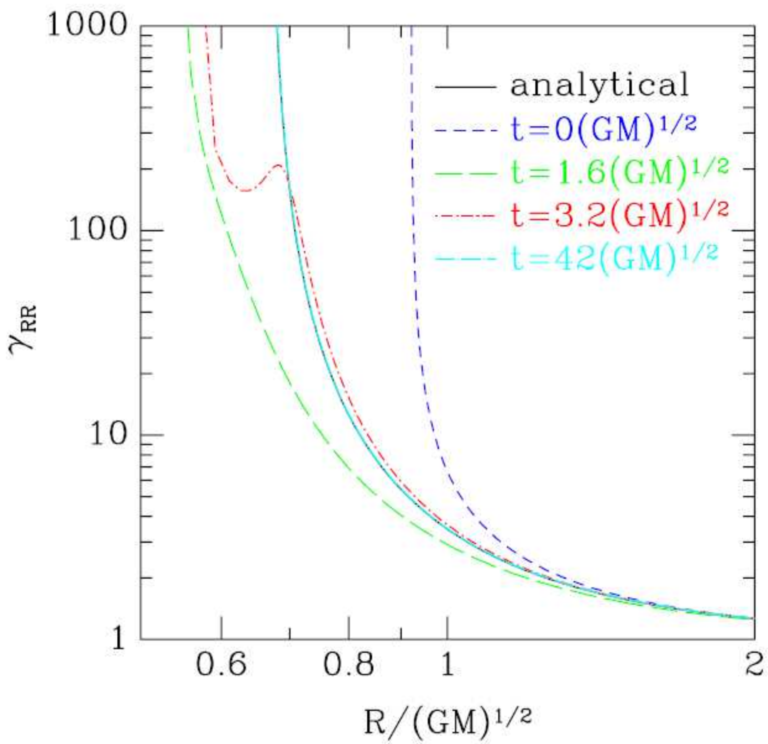
$$\begin{aligned}
\gamma_{ij} &= \chi^{-1}\tilde{\gamma}_{ij}, \quad \det\tilde{\gamma}_{ij} = 1, \\
K_{ij} &= \chi^{-1}\tilde{A}_{ij} + \frac{1}{D-2}\chi^{-1}\gamma_{ij}K, \\
\tilde{\Gamma}^i &= -\tilde{\gamma}^{ij}{}_{,j}.
\end{aligned}$$

$$\begin{aligned}
\left(\partial_t - \beta^i\partial_i\right)\chi &= \frac{2}{D-1}\chi\left(\alpha K - \partial_i\beta^i\right), \\
\left(\partial_t - \beta^l\partial_l\right)\tilde{\gamma}_{ij} &= -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{il}\partial_j\beta^l + \tilde{\gamma}_{jl}\partial_i\beta^l - \frac{2}{D-1}\tilde{\gamma}_{ij}\partial_l\beta^l, \\
\left(\partial_t - \beta^i\partial_i\right)K &= \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{D-1}K^2\right] - D^iD_i\alpha, \\
\left(\partial_t - \beta^l\partial_l\right)\tilde{A}_{ij} &= \chi\left[\alpha\left(\bar{R}_{ij} - \frac{1}{D-1}\gamma_{ij}\bar{R}\right) - \left(D_iD_j\alpha - \frac{1}{D-1}\gamma_{ij}D^lD_l\alpha\right)\right] \\
&\quad + \alpha\left(K\tilde{A}_{ij} - 2\tilde{A}_{il}\tilde{A}^l_j\right) + \tilde{A}_{lj}\partial_i\beta^l + \tilde{A}_{il}\partial_j\beta^l - \frac{D-2}{D-1}\tilde{A}_{ij}\partial_l\beta^l, \\
\left(\partial_t - \beta^j\partial_j\right)\tilde{\Gamma}^i &= \tilde{\gamma}^{jk}\partial_j\partial_k\beta^i + \frac{D-3}{D-1}\tilde{\gamma}^{ij}\partial_j\partial_k\beta^k - \tilde{\Gamma}^j\partial_j\beta^i + \frac{2}{D-1}\tilde{\Gamma}^i\partial_j\beta^j \\
&\quad - 2\tilde{A}^{ij}\partial_j\alpha + 2\alpha\left(\tilde{\Gamma}^i_{jk}\tilde{A}^{jk} - \frac{D-1}{2}\frac{\partial_j\chi}{\chi}\tilde{A}^{ij} - \frac{D-2}{D-1}\tilde{\gamma}^{ij}\partial_jK\right).
\end{aligned}$$

Gauge conditions

Puncture Gauge Conditions Alcubierre+(2003)

$$\begin{aligned}(\partial_t - \beta^i \partial_i)\,\alpha &= -\eta_\alpha \alpha K, \\ (\partial_t - \beta^j \partial_j)\,\beta^i &= -\eta_\beta B^i, \\ (\partial_t - \beta^j \partial_j)\,B^i &= (\partial_t - \beta^j \partial_j)\,\tilde{\Gamma}^i - \eta_B B^i.\end{aligned}$$



Dennison,Wendell,Baumgarte(2010)

η_α, η_β and η_B are arbitrary parameters. (We choose the parameter carefully by problems to solve.)

- we can treat the evolution near the singularity well.
- The result of the evolution of the Scwarzschild-Tangherlini BH: In the puncture gauge, there is a special slice as the attractor. Hannam+(2009),Nakao+(2009)

SACRA code

Yamamoto,Shibata,Taniguchi(2008)

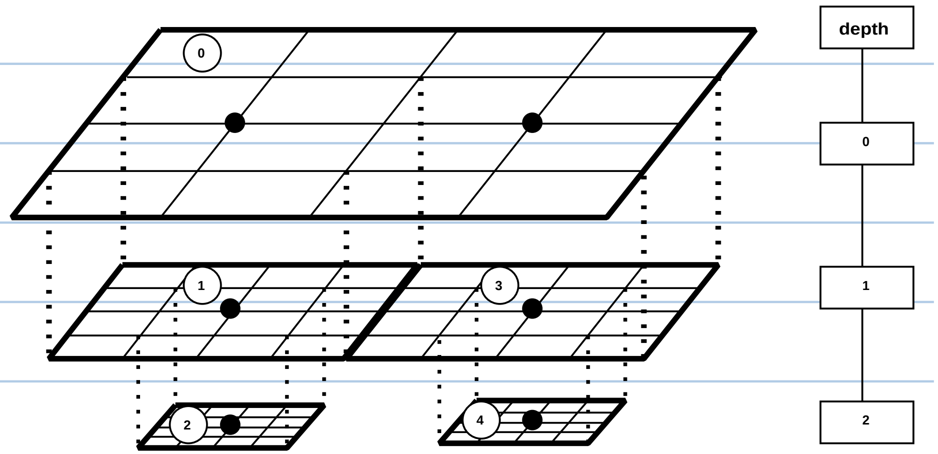
- How to solve the Einstein eq.
  - Space : 4th order finite differences
  - Time : 4th order Runge-Kutta integration
- BH-BH, NS-NS and BH-NS :

  To resolve following scales,
  - Size of the star,
  - Interval between stars,
  - Wavelength of GWs,

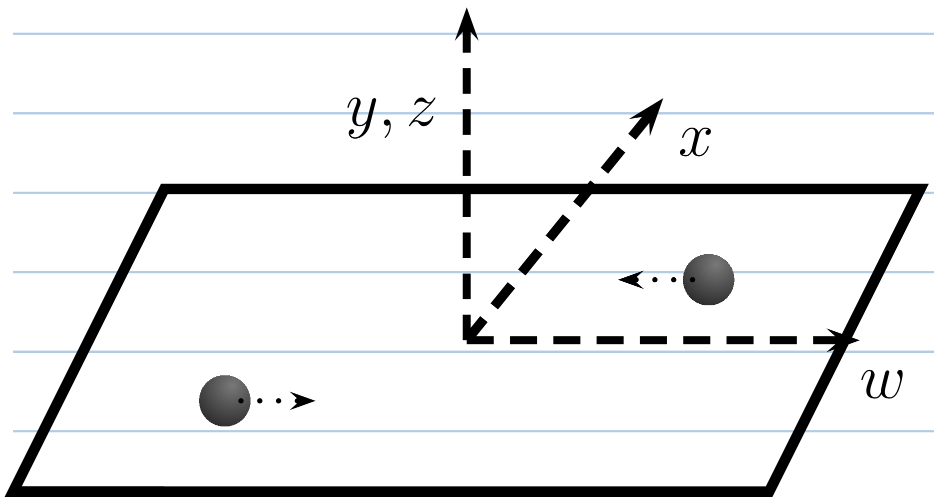
  we need Adaptive Mesh Refinement(AMR).

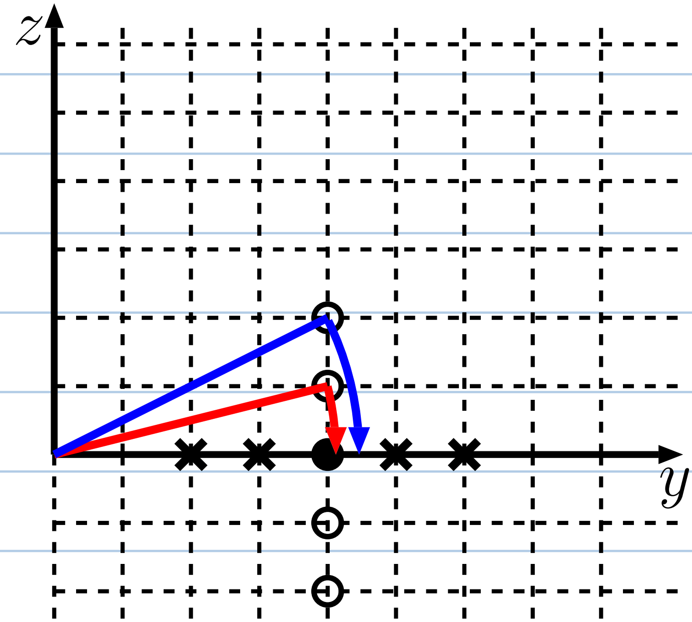
- Economical:

  We can run on a computer,
  - CPU :3.4 GHz, 6 cores,
  - MEM :32 GB,
  - EURO:1,300 euros.



5D Numerical Relativity(Cartoon method)





Cartoon Method Alcubierre+(1999)

- Originally this is used to solve the axisymmetric problems.
- Numerical errors tend to grow near the axis($r \sim 0$) on polar coordinates. (Because $\frac{1}{r}$ diverges apparently at $r = 0$.)
- There isn't such a problem on Cartesian coordinates.

$$\begin{cases} y = a\cos b \\ z = a\sin b \end{cases}$$

We can get any values by the interpolation if we have the data on $z = 0$.

We can take Cartesian coordinates and save the costs by Cartoon method.

Black Hole Collision

Initial condition(a Boost BH)

Schwarzschild-Tangherlini Black Hole in isotropic coordinates

$$ds^2 = -\alpha_0^2 dt_0^2 + \psi_0^{\frac{4}{(D-3)}} \left(dw_0^2 + dx_0^2 + \sum_{i=1}^{D-3} dy_{0i}^2 \right)$$

$$\alpha_0 = \left(1 - \frac{\mu}{4r_0^{D-3}}\right)\psi_0^{-1}, \quad \psi_0 = 1 + \frac{\mu}{4r_0^{D-3}}, \quad r_0^2 = w_0^2 + x_0^2 + \sum_{i=1}^{D-3} y_{0i}^2$$

a Boost BH solution

$$ds^2 = -\gamma^2\left(\alpha_0^2 - v^2\psi_0^{\frac{4}{D-3}}\right)dt^2 - 2\gamma^2 v\left(\psi_0^{\frac{4}{D-3}} - \alpha_0^2\right)dt dw + \psi_0^{\frac{4}{D-3}}\left[B_0^2 dw^2 + dx^2 + \sum_{i=1}^{D-3} dy_i^2\right]$$

$$B_0^2 = \gamma^2\left(1 - \alpha_0^2 v^2 \psi_0^{\frac{-4}{D-3}}\right), \gamma \equiv \frac{1}{\sqrt{1-v^2}}$$

extrinsic curvature

$$K_{xx} = K_{y_i y_i} = \frac{2}{D-3}\frac{\gamma v \alpha_0 \psi_0' w}{\psi_0 B_0 r_0},$$

$$K_{ww} = \frac{\gamma^3 v B_0 w}{r_0}\left[2\alpha_0' - B_0^{-2}\left(\frac{2}{D-3}\alpha_0\psi_0^{-1}\psi_0' - \alpha_0^2\psi_0^{\frac{-4}{D-3}}\alpha_0' v^2\right)\right],$$

$$K_{wx} = \frac{\gamma v B_0 x}{r_0}\left[\alpha_0' - B_0^{-2}\left(\frac{2}{D-3}\alpha_0\psi_0^{-1}\psi_0' - \alpha_0^2\psi_0^{\frac{-4}{D-3}}\alpha_0' v^2\right)\right],$$

$$K_{wy_i} = \frac{\gamma v B_0 y_i}{r_0}\left[\alpha_0' - B_0^{-2}\left(\frac{2}{D-3}\alpha_0\psi_0^{-1}\psi_0' - \alpha_0^2\psi_0^{\frac{-4}{D-3}}\alpha_0' v^2\right)\right].$$

Then, we consider the initial data for BHs with initial velocity by using Boost BHs.

Initial condition(Superposition of Boost BHs)

Initial condition for two Boost BHs

$$\mathrm{d}\ell^2 = \psi^{\frac{4}{D-3}}\left(B^2\mathrm{d}w^2 + \mathrm{d}x^2 + \sum_{i=1}^{D-3}\mathrm{d}y_i^2\right)$$

conformal factor

$$\psi = \psi_{main} + \delta\psi,$$

$$\psi_{main} \equiv 1 + \frac{\mu_1}{4r_1^{D-2}} + + \frac{\mu_2}{4r_2^{D-2}}$$

$$r_A = \sqrt{\gamma^2(w-w_A)^2 + (x-x_A)^2 + \sum_{i=1}^{D-3} y_i^2}$$

$(A = 1, 2)$

$$B^2 = \gamma^2\left[1 - v^2\psi_{main}^{\frac{-4}{D-3}}\left(1 - \frac{\mu_1}{4r_1^{D-3}} - \frac{\mu_2}{4r_2^{D-3}}\right)\right]$$

We should get $\delta\psi$ by solving the Hamiltonian constraint.

extrinsic curvature

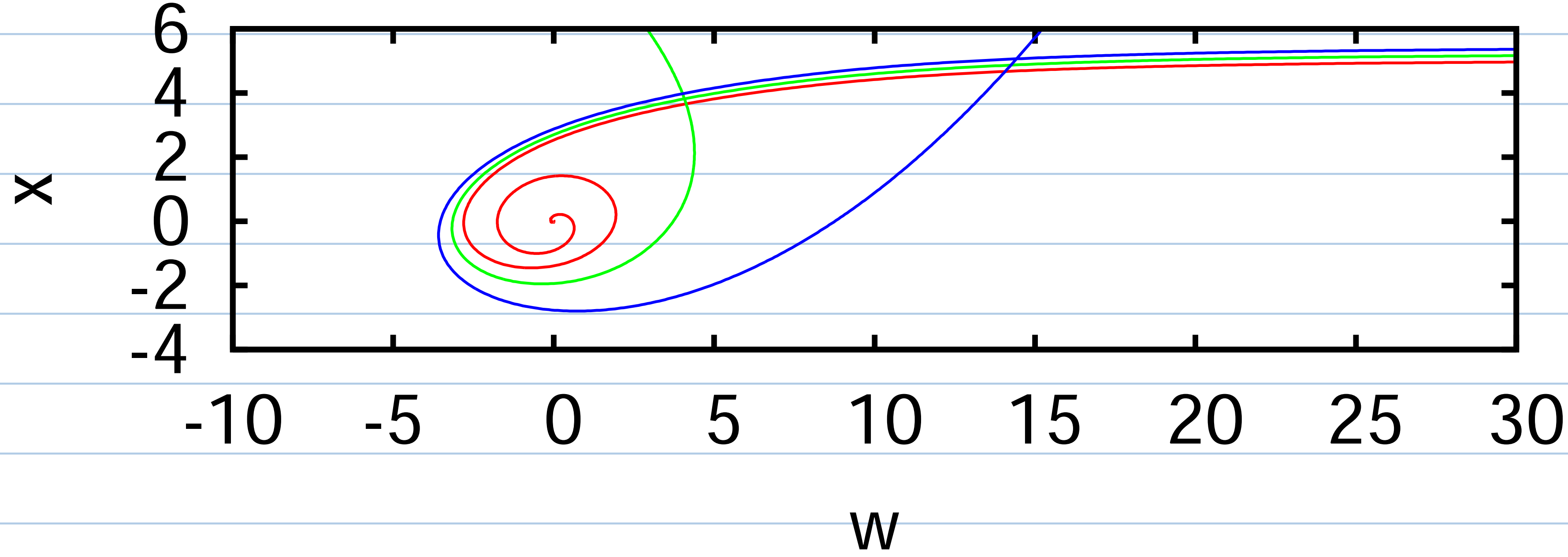
$$K_{ij} = K_{ij}^{(1)} + K_{ij}^{(2)} + \delta K_{ij}$$

We should also get δK_{ij} by solving the Momentum constraints.

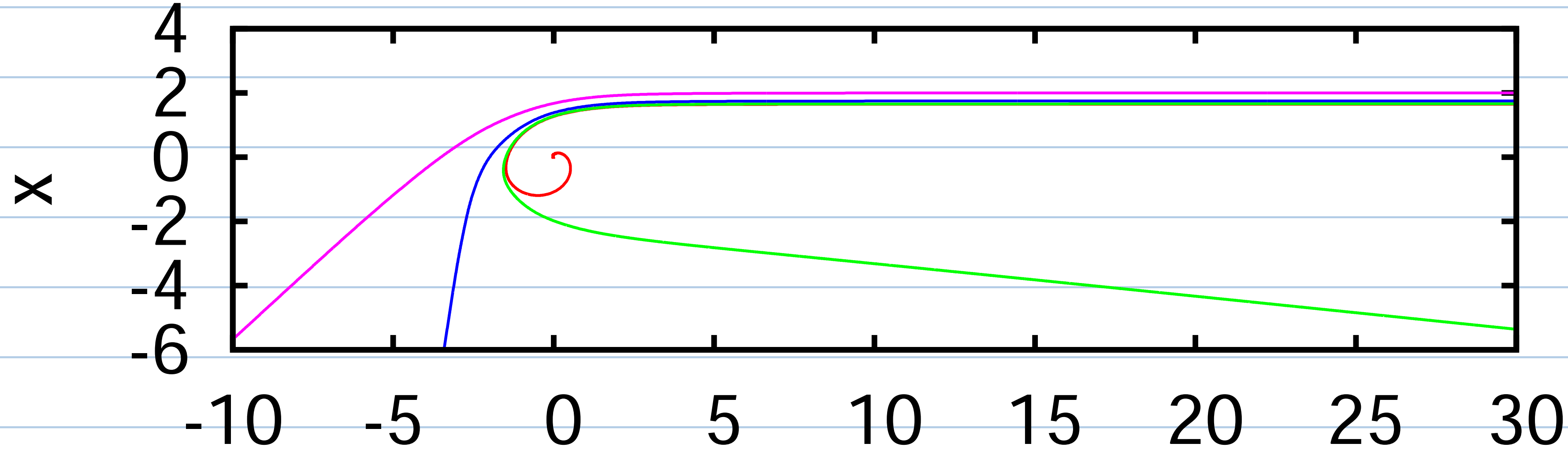
- $\delta\psi$ and δK_{ij} are the small corrections when the BHs are sufficiently apart from each other.

Orbits of BH Collisions in 4D and 5D

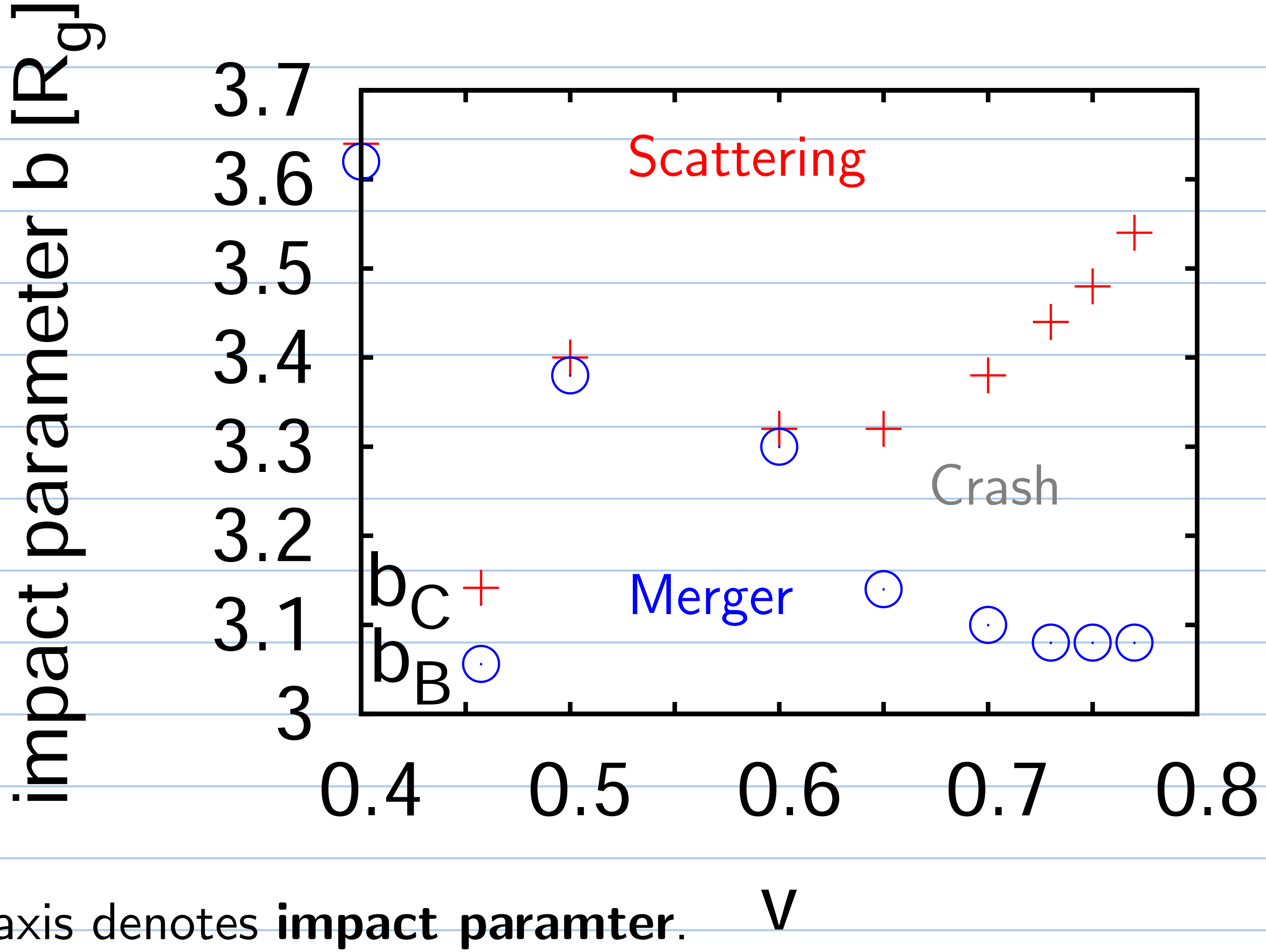
4D $v = 0.6$, Zoom-whirl behavior,impact paramter($x = 5.0, 5.2, 5.4$)



5D $v = 0.6$, No Zoom-whirl, impact paramter($x = 1.65, 1.66, 1.75, 2.0$)



Scattering of BHs

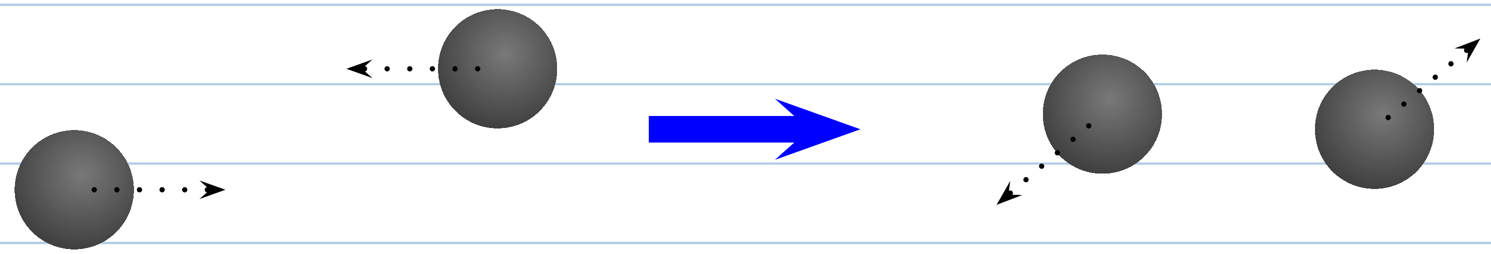


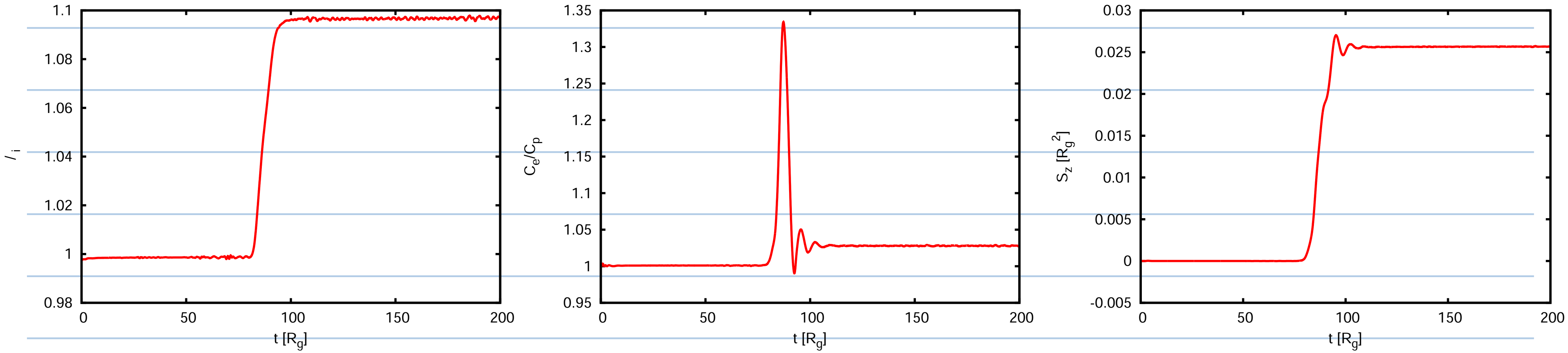
Vertical axis denotes **impact paramter**.
Horizontal axis denotes **initial velocity of BHs**.

b_C:Lower limit of b which we can see the scattering.

b_B:Upper limit of b which we can see the merging.

Exchange of M and J by Scattering





Mass | Deformation of a BH | Spin of a BH

- Mass and Angular momentum grow after scattering.
- "Tidal Heating effect" could explain this.
  (Cf. The membrane paradigm(1986), Poisson(2009,2010))

Kretschmann Invariant Scalar

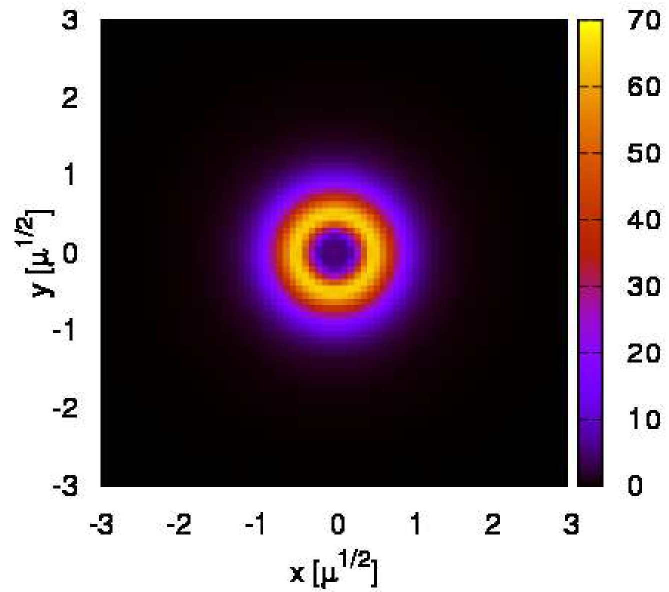
Let's see the BH scattering
in terms of
the Gauge invariant quantity.

$$\boxed{\mathcal{K}^2 = R_{abcd}R^{abcd}}$$
(Kretschmann invariant scalar)

Normalized by the value
on the Horizon of SBH

$$\mathcal{K}^2 = (D-2)^2 (D-1)(D-3) E_P^4$$

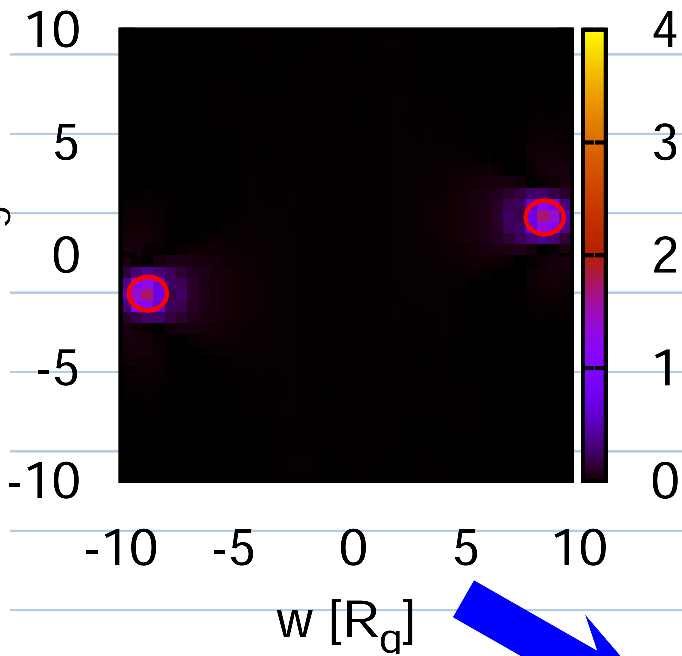
E_P: Planck Energy



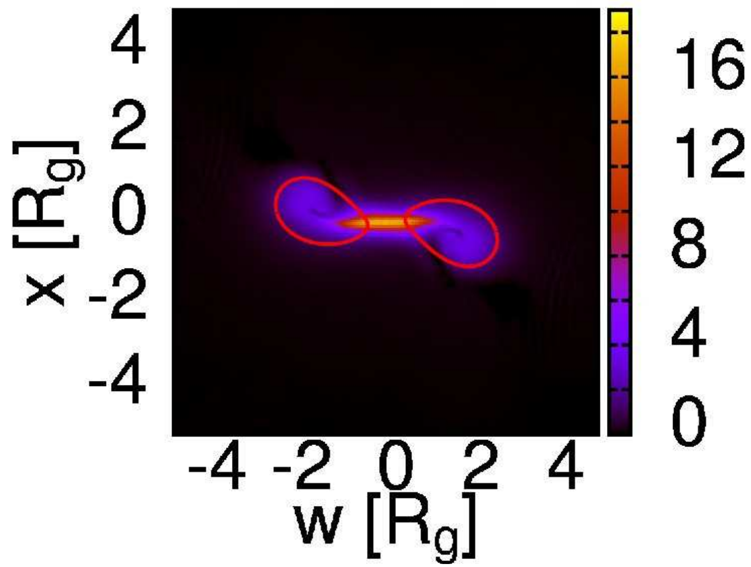
Schwarzschild black hole
in 5D isotropic coordinates

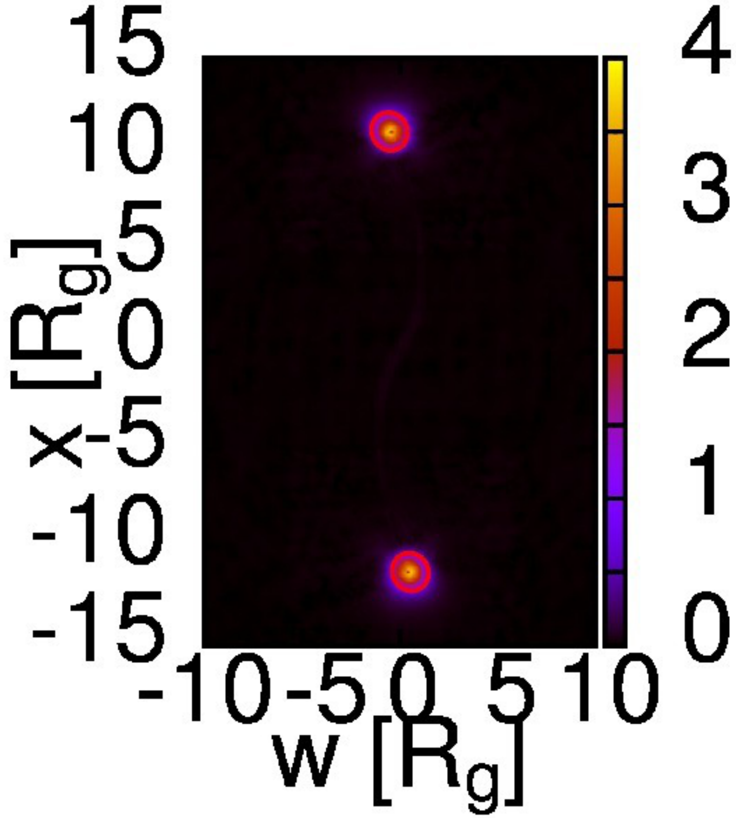
$$\mathcal{K}^2 = 72 E_P^4$$

Kretschmann Scalar during scattering



- Kretschmann invariant $\mathcal{K}$ on $y = z = 0$ plane.
- Red circles denote the shape of AH.

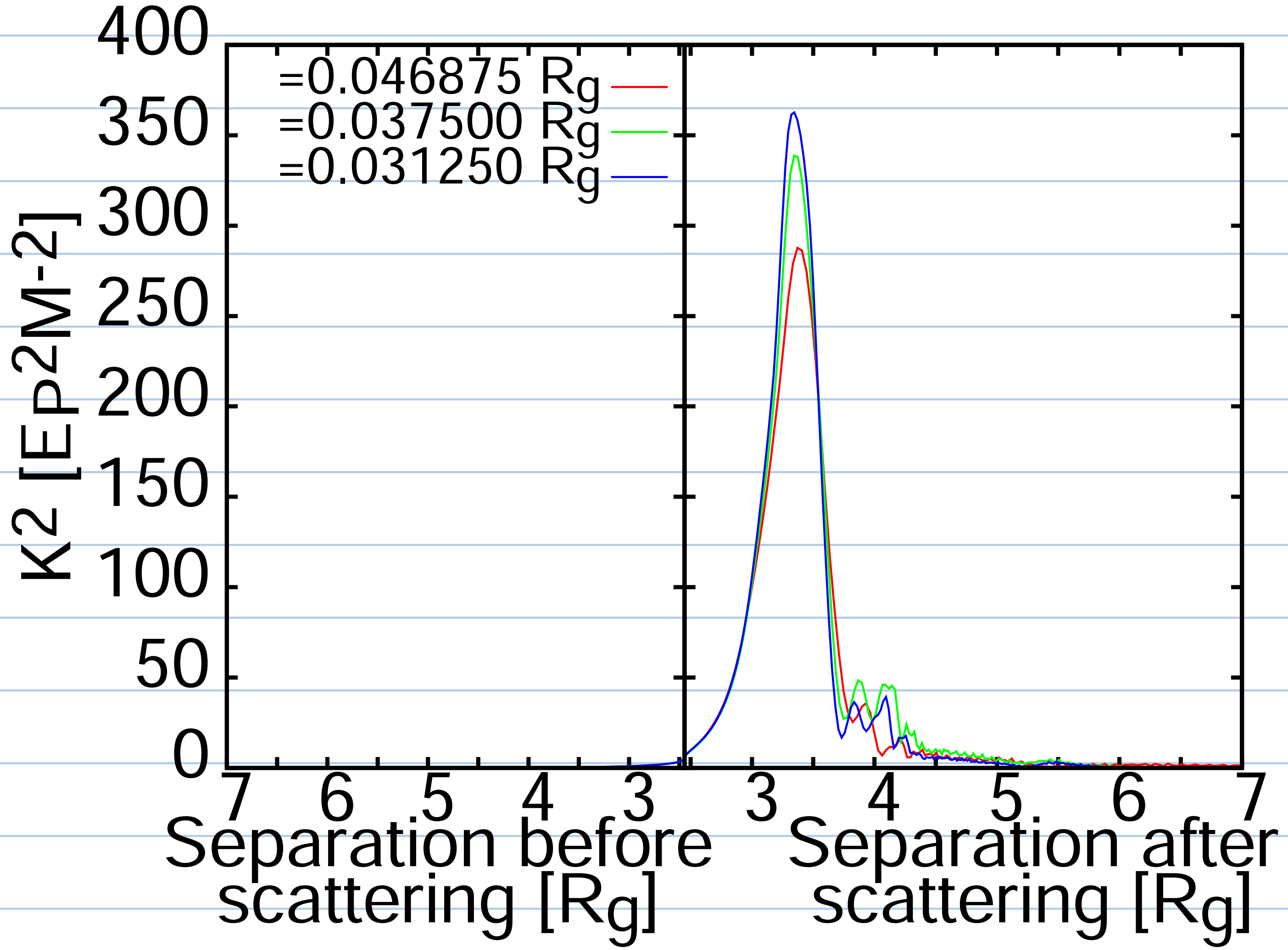




- Two BHs always exist through the scattering.
- Kretschmann becomes large outside the Horizon.

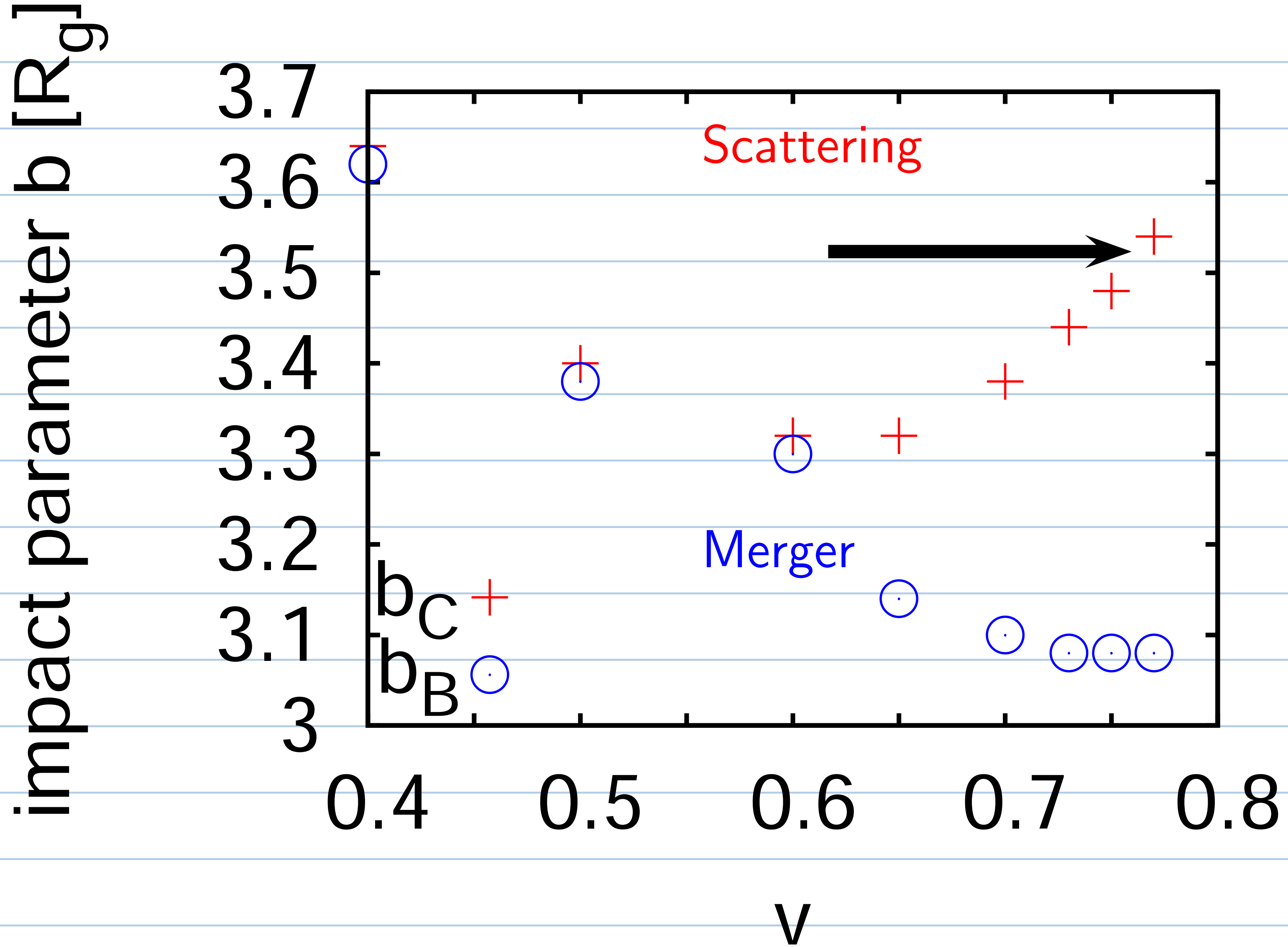
HO, Nakao, Shibata(2011)

Kretschmann vs Separation between BHs



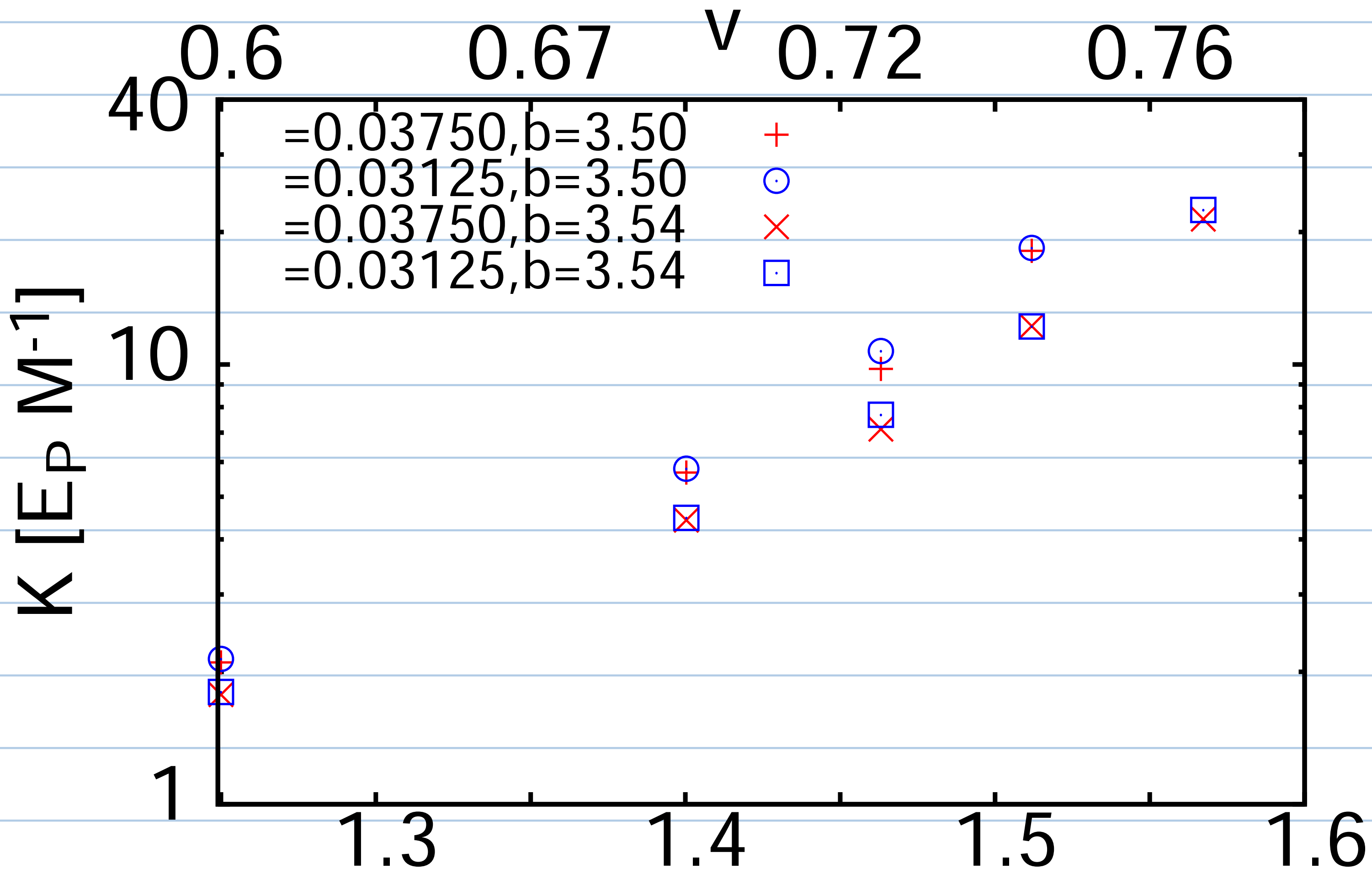
- Horizontal axis denotes **separation between two BHs**. (Time goes from left to right.)
- Vertical axis denotes **Kretchmann scalar** at the center of mass.

Kretschmann vs Initial velocity of BH



Kretschmann invariant at the same impact parameter

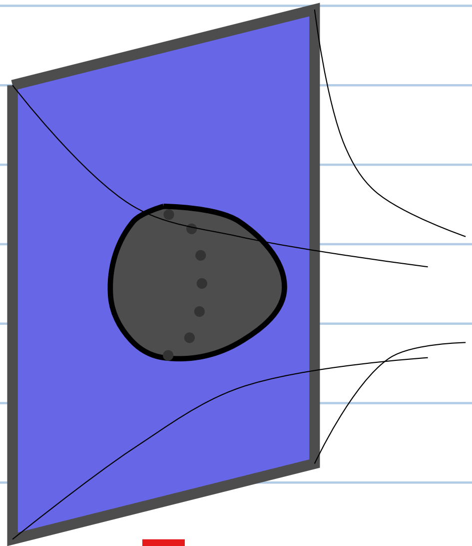
Maximum of Kretschmann Scalar

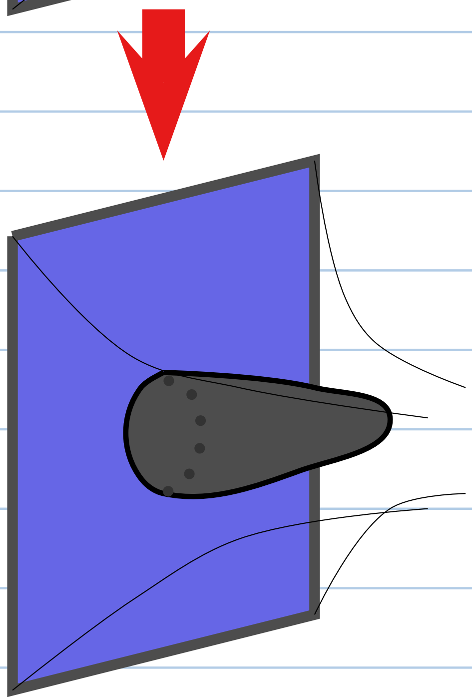


- Horizontal axis denotes the **initial velocity of BHs**.
- Vertical axis denotes the **Maximum of** $\mathcal{K}$ at the center of mass(log scale).

Black Hole in AdS spacetimes

BH in the spacetimes with a brane





Future Prediction?

Looking back at the history briefly,

L. Randall and R. Sundrum (1999)
Warped spacetimes

T. Tanaka(2002), R. Emparan *et al.*(2002)
BHs larger than bulk scale are not static.
H. Kudoh, T. Tanaka and T. Nakamura(2003)
Method to make the solution with a BH. (Small BH could be static.)

H. Kudoh(2004), H. Yoshino(2006)
It is difficult to make a large BH.

N. Tanahashi and T. Tanaka(2008)
A larger BH seems unstable.

P. Figueras and T. Wiseman(2011)
Method to make a large BH.

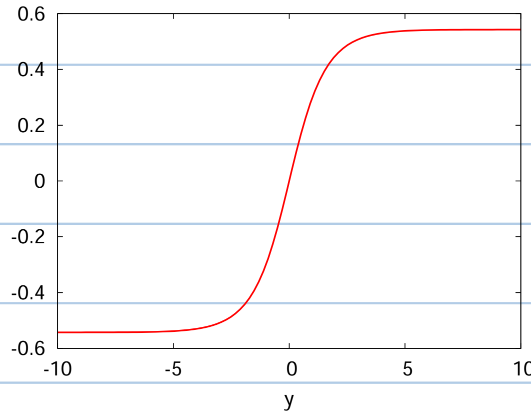
We want to see **Dynamical Evolution**.

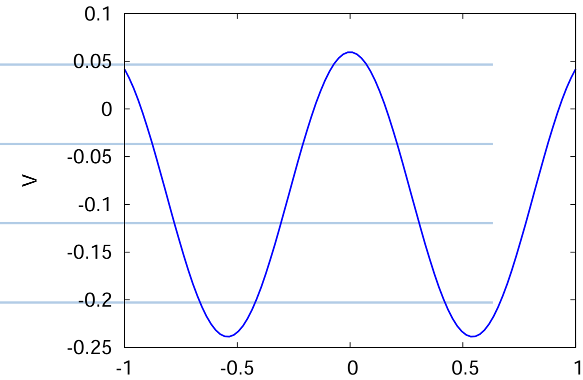
Exact Solution of Thick Brane

T. Takahashi, HO, M.Shibata(2012-13?)

(My) Requirement

- I want to start from SACRA code.
- I like easy boundary conditions.





We can make the static thick brane with a scalar field.(Cf. Giovannini(2002))

$$S = \int d^5x\sqrt{-g}\left[\frac{R}{2\kappa} - \frac{1}{2}\nabla_a\phi\nabla^a\phi - V(\phi)\right] \quad (1)$$

$$\phi(y) = \sqrt{\frac{3}{\kappa}}\arctan\left[\sinh(by)\right], \quad z = 2b\sinh(by) \quad (2)$$

$$V(\phi) = \frac{3b^2}{2\kappa}\left[1 - 5\sin^2(\sqrt{\frac{\kappa}{3}}\phi)\right] \quad (3)$$

Metric of the Brane Solution (Cf. RS II model)

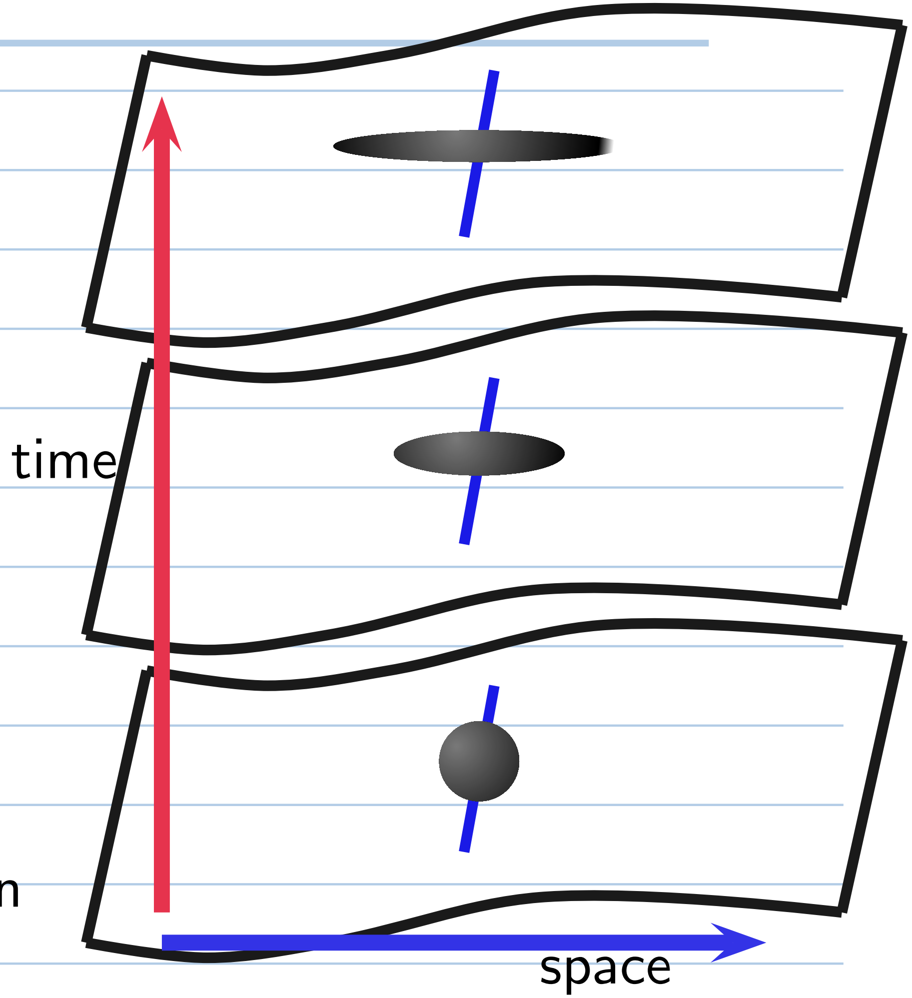
$$ds^2 = \frac{\mathbf{1}}{\mathbf{1+b^2z^2}}\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right] \quad (4) \qquad ds^2 = \frac{\ell^2}{z^2+\ell^2}\left[dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right]$$

$b = \ell^{-1}$ is the thickness of the brane.

5D NR(4+1 decomposition)

To solve Einstein equation

$$\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = \kappa T_{ab}$$



Spatial metric $\gamma_{ab} \equiv g_{ab} + n_a n_b$

n^a:Normal vector($n^a = (1/\alpha, \beta^i/\alpha)$)

Extrinsic curvature

$$K_{ab} \equiv -\frac{1}{2\alpha}\left(\partial_t \gamma_{ab} - D_b \beta_a - D_a \beta_b\right)$$

Decomposition of the Einstein equation

$\longrightarrow$ **Same form as usual**

Evolution equation

$$\partial_t K_{ab} = \beta^c D_c K_{ab} + K_{cb} D_a \beta^c + K_{ca} D_b \beta^c$$
$$+\alpha\left(R_{ab} - 2K_{ac}K^c_b + K_{ab}K - \kappa\left[S_{ab} + \frac{\rho - S}{3}\gamma_{ab}\right]\right)$$
$$-D_b D_a \alpha$$

Constraints

Hamiltonian constraint

$$R + K^2 - K_{ab}K^{ab} = 2\kappa\rho$$

Momentum constraints

$$D_b K^b_a - D_a K = \kappa j_a$$

BSSN formalism in AdS spacetimes

$$\mathrm{d}s^2 = \frac{1}{a^2(z)}\left[-\alpha^2\mathrm{d}t^2 + \gamma_{ij}\left(\mathrm{d}x^i + \beta^i\mathrm{d}t\right)\left(\mathrm{d}x^j + \beta^j\mathrm{d}t\right)\right] \quad (5)$$

- We define the BSSN variables without $a(z)$ part.
- We get usual evolution equation with BSSN variables + $a(z)$ terms.

$$\left(\partial_t - \beta^i\partial_i\right)\chi = \frac{1}{2}\chi\left(\alpha K - \partial_i\beta^i\right) + \frac{2a'}{a}\beta^z\chi,$$

$$\left(\partial_t - \beta^l\partial_l\right)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{il}\partial_j\beta^l + \tilde{\gamma}_{jl}\partial_i\beta^l - \frac{1}{2}\tilde{\gamma}_{ij}\partial_l\beta^l,$$

$$\left(\partial_t - \beta^i\partial_i\right)K = \alpha\left[\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{4}K^2 + \frac{\kappa}{3}(2\rho + S)\right] - \left[a\bar{D}^i\bar{D}_i\left(\frac{\alpha}{a}\right)\right]^{TF} + \frac{a'}{a}\beta^z K,$$

$$\left(\partial_t - \beta^l\partial_l\right)\tilde{A}_{ij} = \cdots,$$

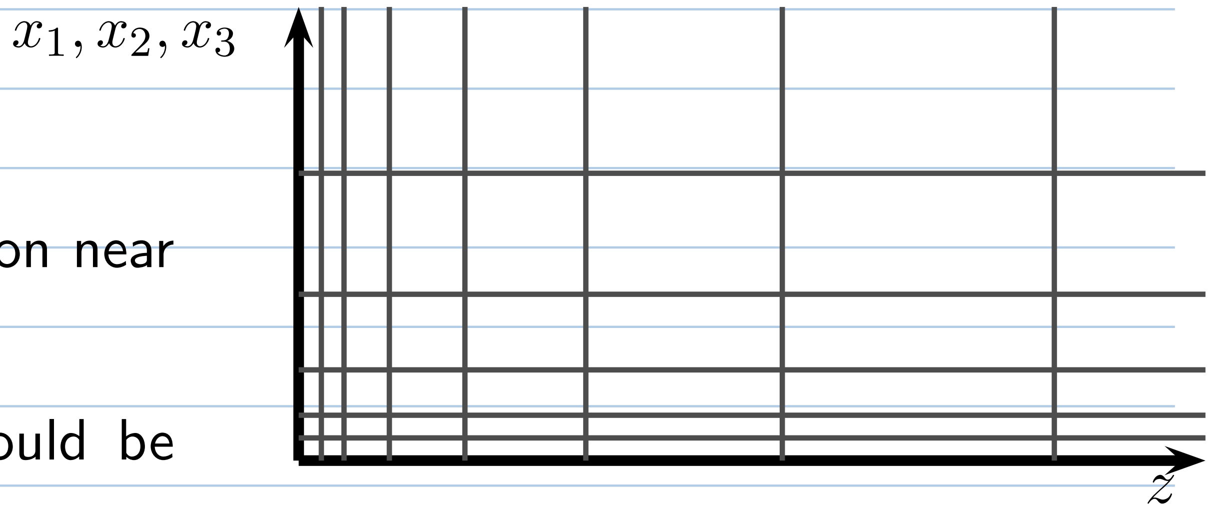
$$\left(\partial_t - \beta^j\partial_j\right)\tilde{\Gamma}^i = \cdots$$

Evolution equations for the scalar field

$$\left(\partial_t - \beta^l\partial_l\right)\phi = -\alpha\Pi,$$

$$\left(\partial_t - \beta^l\partial_l\right)\Pi = \alpha K\Pi + \frac{\alpha}{a^2}\frac{\partial V}{\partial\phi} - \chi\alpha\left[\tilde{D}^i\tilde{D}_i\phi + \left(\frac{\partial_i\alpha}{\alpha} - \frac{\partial\chi}{\chi} - \frac{3a'}{a}\delta^z_i\right)\tilde{D}^i\phi\right].$$

Coordinates and Boundary Conditions



Coordinates

- We want higher resolution near the brane.
- Perhaps outer region could be coarse.(not sure)
- We use y coordinate.
  (The interval of z is larger at far.)
  $z = 2b\sinh(by)$
- But we defined BSSN variables on z(γ_{zz} conformal flat).
- We should pay attention to the derivatives of z.
  $\partial_z \alpha = \frac{1}{a(y)} \partial_y \alpha$

Boundary Conditions

- Z_2 symmetry at the brane
  We can give the variables at $-y$ from that at y.
  $\alpha(-y) = \alpha(y), \phi(-y) = -\phi(y)$
- If outer boundaries are enough far, we can impose the Neumann condition or Outgoing condition. (It could be correct unless we evolve for a long time.)

Hamiltonian Constraint

- We want an initial condition with Apparent Horizon(AH).
- Previous study T. Shiromizu and M. Shibata(2000)
- They construct the solution with AH by introducing a scalar field as gravitational source.

metric ansatz

$$dl^2 = \frac{1}{a^2(z)}\left[dz^2 + (1+w(r,z))^4\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)\right]$$

Hamiltonian Constraint

$$\partial_r^2 w + \frac{2}{r}\partial_r w + \frac{3}{2}\left[\left(\partial_z^2 w - \frac{3bz^2}{1+b^2z^2}\partial_z w\right)(1+w)^4 + 3(\partial_z w)^2(1+w)^3\right] = -\pi(1+w)^5(\partial_z\psi)^2 - \pi(1+w)(\partial_r\psi)^2$$

We can solve the nonlinear elliptic equation(in principle if solutions exist)!

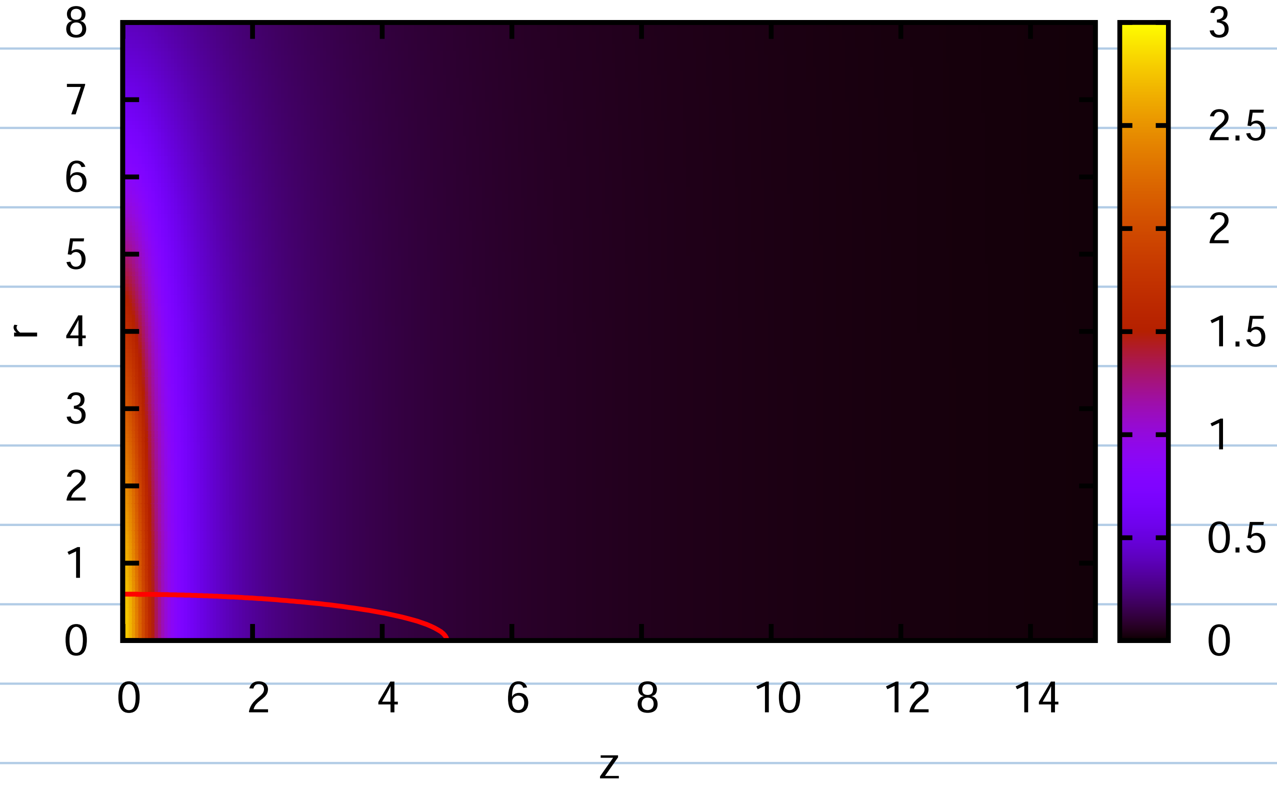
Initial Condition

How to give the scalar field:

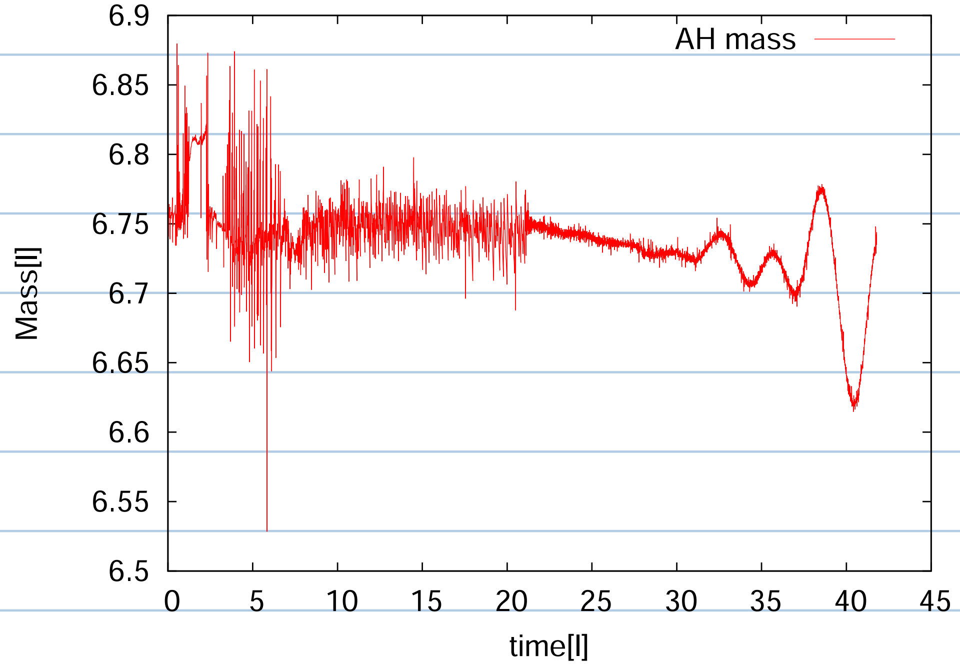
$$\psi(r,z) = \frac{f(r,z)}{(1+w)^s}$$

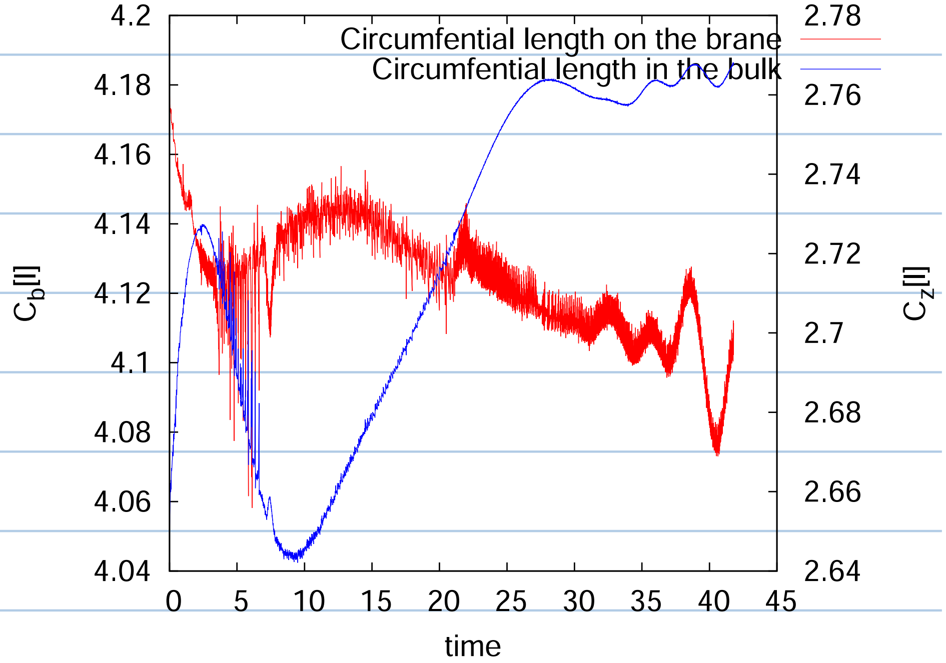
$$f(r,z) = \epsilon z \exp\left\{-\frac{1}{2}\left(\frac{r^2}{a^2}+\frac{z^2}{b^2}\right)\right\}$$

We can choose s, a, b and ϵ.
for example,
$a = b = 0.3, s = 2, \epsilon = 2.25$



Preliminary Result





- This is an evolution test for previous initial condition(Low res.).
- The values related to the AH are in about 4% error.
- It seems good before $t \sim 20$.
- The proper length of the equator is not so changed, but the position of AH on those coordinates is growing.

Summary

- We are in the stage to apply Numerical relativity to various space-times.
- We can see the dynamical evolution of the BH(s) in higher dimensions.
- Scattering of BHs in higher dimensions might be used to investigate the limit of classical gravity.
- The code of the AdS numerical relativity should show some interesting results(I hope).

Thank you very much for listening!