Turbulent Instability of Anti-de Sitter Space?

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Outline

- Motivation: to get some hints on the stability of a perturbed AdS space (what happens if perturbation can not disperse?)
- Model: Self-gravitating massless scalar field in $3+1$ at spherical symmetry, with $\Lambda < 0$
  (Choptuik 1993: $3+1$, $\Lambda = 0$)
  (Pretorius, Choptuik 2000: $2+1$, $\Lambda < 0$)
- Results
- Other models
- Final remarks

Anti-de Sitter spacetime in $d+1$ dimensions

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{1+r^2/\ell^2} + r^2 d\Omega^2_{S^{d-1}}, \qquad 0 \le r, \;\; -\infty < t < \infty$$

is a maximally symmetric solution of vacuum Einstein equations with a negative cosmological constant $\Lambda = -d(d-1)/(2\ell^2)$:

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} + \Lambda\, g_{\alpha\beta} = 0\,.$$

Set $r/\ell = \tan(\rho/\ell)$ to get

$$ds^2 = \frac{1}{\left(\cos\frac{\rho}{\ell}\right)^2}\left[-dt^2 + d\rho^2 + \ell^2\left(\sin\frac{\rho}{\ell}\right)^2 d\Omega^2_{S^{d-1}}\right],$$

$$-\infty < t < +\infty, \quad 0 \le \rho/\ell < \pi/2\,.$$

Conformal infinity $\rho/\ell = \pi/2$ is the timelike surface $\mathcal{I} = \mathbb{R} \times S^{d-1}$ with the boundary metric $ds^2_{\mathcal{I}} = -dt^2 + d\Omega^2_{S^{d-1}}$

Maximally symmetric solutions of vacuum Einstein's equations and their stability

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} + \Lambda\, g_{\alpha\beta} = 0\,.$$

- $\Lambda = 0$: Minkowski (trivial, yet most important)
  asymptotically stable (Christodoulou&Klainerman 1993),
- $\Lambda > 0$: de Sitter (important in cosmology - Nobel Prize 2011)
  asymptotically stable (Friedrich 1986),
- $\Lambda < 0$: anti- de Sitter (most popular on arXiv due to AdS/CFT)

Is AdS stable?

- A solution (of a dynamical system) is said to be stable if small perturbations of it at $t = 0$ remain small for all later times
- Linearly stable, asymptotic stability precluded (can not relax if perturbed); stability - an open problem (first addressed by Anderson 2005).
- Key difference between Minkowski and AdS: the main mechanism of stability of Minkowski - dissipation of energy by dispersion - is absent in AdS because AdS is effectively bounded (for no flux boundary conditions at $\mathcal{I}$ it acts as a perfect cavity)
- Note that by positive energy theorems both Minkowski and AdS are the unique ground states among asymptotically flat/AdS spacetimes

Model

- To make the problem feasible we start with spherical symmetry (effectually $1+1$ dimensional problem)
- Spherically symmetric vacuum solutions are static (Birkhoff's theorem) $\Rightarrow$ we need matter to generate dynamics
- Simple matter model: massless scalar field ϕ in 3+1 dimensions

$$G_{\alpha\beta} + \Lambda\, g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi\, \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} \partial_\mu \phi \partial^\mu \phi \right), \qquad \Lambda = -3/\ell^2\,,$$

$$g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi = 0$$

- In the corresponding asymptotically flat ($\Lambda = 0$) model Christodoulou proved the weak cosmic censorship (dispersion for small data and collapse to a black hole for large data) and Choptuik discovered critical phenomena at the threshold for black hole formation
- Remark: For even $d \geq 4$ there is a way to bypass Birkhoff's theorem (cohomogeneity-two Bianchi IX ansatz, Bizoń, Chmaj, Schmidt 2005)

- Convenient parametrization of asymptotically AdS spacetimes

$$ds^2 = \frac{1}{\left(\cos\frac{\rho}{\ell}\right)^2}\left[-Ae^{-2\delta}dt^2 + A^{-1}d\rho^2 + \ell^2\left(\sin\frac{\rho}{\ell}\right)^2 d\Omega_2^2\right].$$

where A and δ are functions of (t,ρ).

- Auxiliary variables $\Phi = \phi'$ and $\Pi = A^{-1}e^{\delta}\dot{\phi}$ $\quad (' = \partial_\rho, \dot{} = \partial_t)$
- Field equations (using units where $4\pi G = 1$)

$$A' = (1-A)\frac{1+2\left(\sin\frac{\rho}{\ell}\right)^2}{\ell\left(\cos\frac{\rho}{\ell}\right)\left(\sin\frac{\rho}{\ell}\right)} - \ell\left(\cos\frac{\rho}{\ell}\right)\left(\sin\frac{\rho}{\ell}\right)A\left(\Phi^2+P^2\right),$$

$$\delta' = -\ell\left(\cos\frac{\rho}{\ell}\right)\left(\sin\frac{\rho}{\ell}\right)\left(\Phi^2+P^2\right),$$

$$\dot{\Phi} = \left(Ae^{-\delta}P\right)', \qquad \dot{P} = \frac{1}{\left(\tan\frac{\rho}{\ell}\right)^2}\left[\left(\tan\frac{\rho}{\ell}\right)^2 Ae^{-\delta}\Phi\right]'.$$

- AdS space: $\phi \equiv 0$, $A \equiv 1$, $\delta \equiv 0$; now we want to solve the initial-boundary value problem for this system for small perturbation generated with some small, smooth initial data $(\phi, \dot{\phi})_{|t=0}$

Boundary conditions

- We assume that initial data $(\phi, \dot\phi)_{|t=0}$ are smooth
- Smoothness at the center implies that near $\rho = 0$

$$\phi(t,\rho) = f_0(t) + \mathcal{O}(\rho^2), \quad \delta(t,\rho) = \mathcal{O}(\rho^2), \quad A(t,\rho) = 1 + \mathcal{O}(\rho^2)$$

- Smoothness at spatial infinity and conservation of the total mass M imply that near $\rho = \ell\pi/2$ (using $\xi = \pi/2 - \rho/\ell$)

$$\phi(t,\rho) = f_\infty(t)\,\xi^3 + \mathcal{O}\left(\xi^5\right), \quad \delta(t,\rho) = \delta_\infty(t) + \mathcal{O}\left(\xi^6\right),$$
$$A(t,\rho) = 1 - 2(M/\ell)\xi^3 + \mathcal{O}\left(\xi^6\right)$$

  Remark: There is no freedom in prescribing boundary data
- Local well-posedness (Friedrich 1995, Holzegel&Smulevici 2011)
- mass function and asymptotic mass:

$$m(t,\rho) = \frac{\ell \sin(\rho/\ell)}{2}\,\frac{1 - A(t,\rho)}{\cos^3(\rho/\ell)}$$

$$M = \lim_{\rho\to\pi\ell/2} m(t,\rho) = \frac{1}{2}\int_0^{\pi\ell/2} \left(A\Phi^2 + A\Pi^2\right)\left(\tan\frac{\rho}{\ell}\right)^2 d\rho$$

Reminder: asymptotically flat ($\Lambda = 0$) self-gravitating scalar field

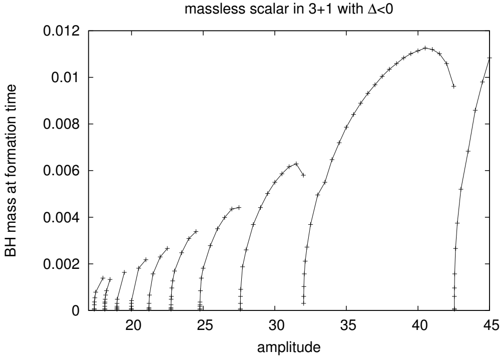
- Christodoulou (1986-1993): dispersion for small data and collapse to a black hole for large data (proof of the weak cosmic censorship)
- Consider a family of initial data $\Phi(p)$ which interpolates between dispersion and collapse (Choptuik 1993)
- There exists a critical value of the parameter p^* such that
  - $p < p^* \Rightarrow$ dispersion
  - $p > p^* \Rightarrow$ black hole
- Universal behavior in the near-critical region $|p - p^*| \ll 1$
  - $m_{BH} \sim (p^* - p)^\gamma$ with universal exponent γ
  - discretely self-similar attractor with universal period Δ
- Critical solution ($p = p^*$) is a non-generic naked singularity

Realplayer

Critical behavior

Initial data: $\Phi(0,x) = 0\,, \Pi(0,x) = \varepsilon\left[\exp\left(-\frac{\ell\tan(\rho/\ell)}{\sigma}\right)^2\right]$

We fix $\sigma = 1/16$ and vary ε.



BH mass vs. amplitude

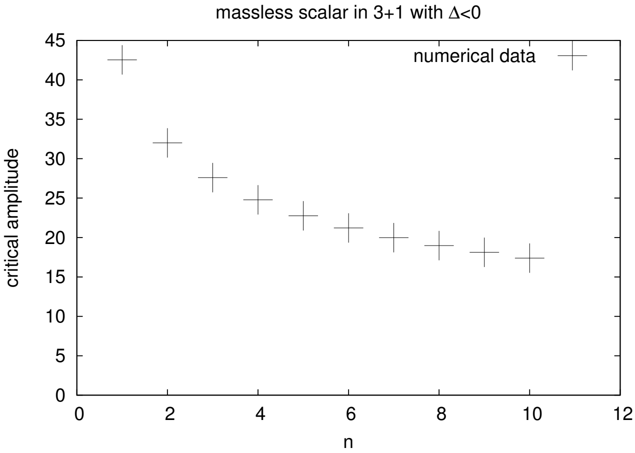
There is a decreasing sequence of critical amplitudes ε_n for which the evolution, after making n reflections from the AdS boundary, locally asymptotes Choptuik's solution. In each small right neighborhood of ε_n

$$m_{BH}(\varepsilon) \sim (\varepsilon - \varepsilon_n)^\gamma$$

with $\gamma \simeq 0.37$. It seems that $\lim_{n\to\infty} \varepsilon_n = 0$

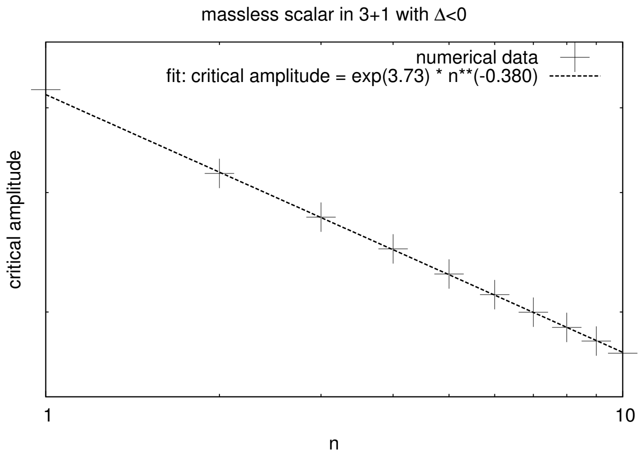
Remark: The generic endstate of evolution is the Schwarzschild-AdS BH of mass M (in accord with Holzegel&Smulevici 2011)

The sequence of critical amplitudes (1)



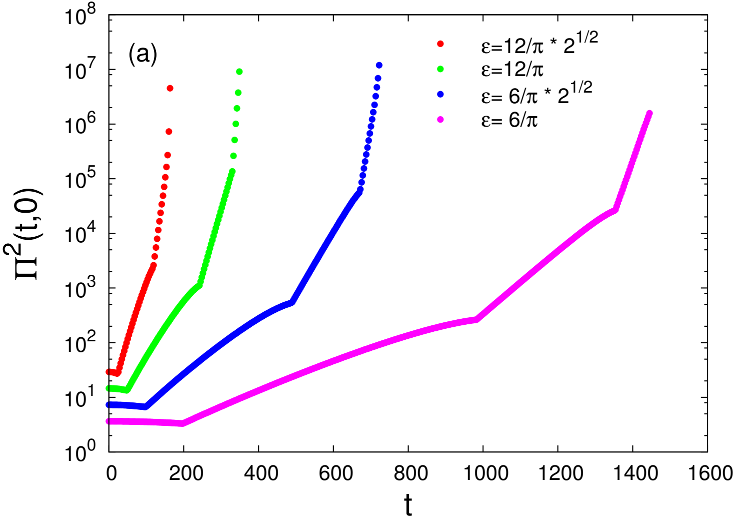
- Does this sequence go to zero ?
- Is it $\sim n^{-\alpha}$ dependence ?

The sequence of critical amplitudes (2)



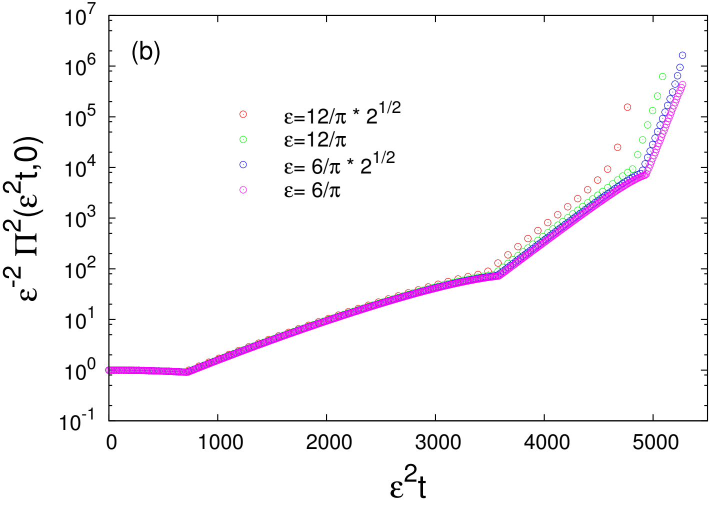
- a hint for the instability of AdS

Key evidence for instability



Ricci scalar $R = 2\left(\Phi^2 - \Pi^2\right)/\ell^2 - 12/\ell^2$

Key evidence for instability



Onset of instability at time $t = \mathcal{O}(\varepsilon^{-2})$

Weakly nonlinear perturbations

- From now $\rho/\ell \equiv x$
- We seek an approximate solution starting from small initial data $(\phi, \dot{\phi})_{|t=0} = (\varepsilon f(x), \varepsilon g(x))$
- Perturbation series

$$\phi = \varepsilon\phi_1 + \varepsilon^3\phi_3 + \ldots$$
$$\delta = \varepsilon^2\delta_2 + \varepsilon^4\delta_4 + \ldots$$
$$1 - A = \varepsilon^2 A_2 + \varepsilon^4 A_4 + \ldots$$

where $(\phi_1, \dot{\phi}_1)_{|t=0} = (f(x), g(x))$ and $(\phi_j, \dot{\phi}_j)_{|t=0} = (0, 0)$ for $j > 1$.

- Inserting this expansion into the field equations and collecting terms of the same order in ε, we obtain a hierarchy of linear equations which can be solved order-by-order.

First order

- Linearized equation (Ishibashi&Wald 2004)

$$\ddot{\phi}_1 + L\phi_1 = 0, \quad L = -\frac{1}{\tan^2 x}\,\partial_x\left(\tan^2 x\,\partial_x\right)$$

The operator L is essentially self-adjoint on $L^2([0,\pi/2), \tan^2 x\, dx)$.

- Eigenvalues and eigenvectors of L are $(j = 0, 1, \dots)$

$$\omega_j^2 = (3+2j)^2, \quad e_j(x) = d_j\,(\cos x)^3\, F\left(\begin{array}{c} -j,\ 3+j \\ 3/2 \end{array}\middle|\,(\sin x)^2\right)$$

$\Rightarrow$ AdS is linearly stable

- Linearized solution

$$\phi_1(t,x) = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j)\, e_j(x)$$

where amplitudes a_j and phases β_j are determined by the initial data.

Second order (back-reaction on the metric)

$$A_2' + \frac{1 + 2\sin^2 x}{\sin x \cos x} A_2 = \sin x \cos x \left(\dot{\phi}_1^2 + {\phi_1'}^2\right)$$
$$\delta_2' = -\sin x \cos x \left(\dot{\phi}_1^2 + {\phi_1'}^2\right)$$

so

$$A_2(t,x) = \frac{\cos^3 x}{\sin x} \int_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) \tan^2 y \, dy$$

$$\delta_2(t,x) = -\int_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) \sin y \cos y \, dy$$

It follows that

$$M = \frac{\varepsilon^2}{2} \int_0^{\pi/2} \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2\right) \tan^2 y \, dy + \mathcal{O}(\varepsilon^4)$$

Third order

- $\ddot{\phi}_3 + L\phi_3 = S(\phi_1, A_2, \delta_2)\,,$ $(\star)$
  where $S := 2(A_2+\delta_2)\ddot{\phi}_1 + (\dot{A}_2+\dot{\delta}_2)\dot{\phi}_1 + (A_2'+\delta_2')\phi_1'.$

- Projecting Eq.$(\star)$ on the basis $\{e_j\}$ we obtain an infinite set of decoupled forced harmonic oscillations for the generalized Fourier coefficients $c_j(t) := (e_j,\, \phi_3)$

$$\ddot{c}_j + \omega_j^2 c_j = S_j := (e_j,\, S) \quad \text{and} \quad (c_j, \dot{c}_j)\big|_{t=0} = 0$$

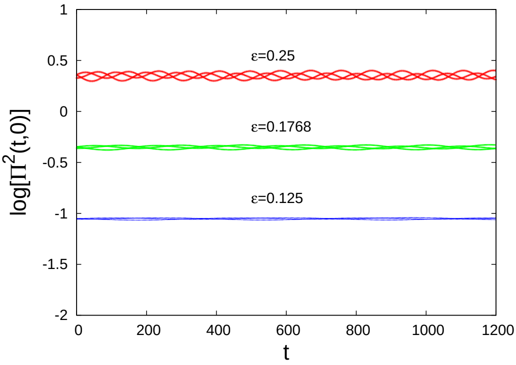
- Let Ω_1 be a set of frequencies entering the linearized solution $\phi_1(t,x) = \sum_j \left[\omega_j \in \Omega_1\right] a_j \cos(\omega_j t + \beta_j)\, e_j(x)$.
  A not-quite-straightforward calculation yields that for each j such that $\omega_j = |\omega_1 + \omega_2 - \omega_3|$, $\omega_i \in \Omega_1$ there is a resonant term in S_j (i.e., a term proportional to $\cos\omega_j t$ or $\sin\omega_j t$). Such term gives rise to a secular term in c_j, that is $c_j \sim t \sin\omega_j t$ or $c_j \sim t\cos\omega_j t$. Some of these resonances result in frequency shift and are harmless for stability, but the others put stability in question!

Example 1: single-mode data $\phi(0,x) = \varepsilon\, e_0(x)$

- First order $\phi_1(t,x) = \cos(\omega_0 t) e_0(x), \quad \omega_0 = 3 \qquad (\omega_j = 3 + 2j)$
- Third order $\phi_3(t,x) = \sum_{j=0}^{\infty} c_j(t) e_j(x), \quad (c_j, \dot{c}_j)|_{t=0} = 0$ and

$$\ddot{c}_j + \omega_j^2 c_j = b_{j,0} \cos(\omega_0 t) + b_{j,3} \cos(\omega_3 t).$$

But $b_{3,3} = 0$ (!) and only $j = 0$ is resonant.



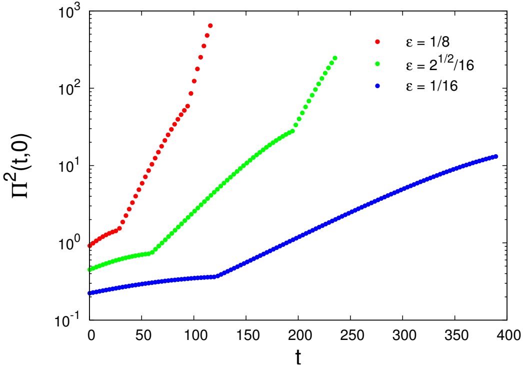
The $j = 0$ resonance can be easily removed by the two-scale method (slow-time phase modulation) which gives $\phi \simeq \varepsilon \cos(3t + \frac{153}{4\pi}\varepsilon^2 t)\, e_0(x)$. This suggests that there are non-generic initial data which may stay close to AdS solution

Example 2: two-mode data $\phi(0,x) = \varepsilon\,(e_0(x) + e_1(x))$

- First order $\phi_1(t,x) = \cos(\omega_0 t)e_0(x) + \cos(\omega_1 t)e_1(x)$, $\omega_0 = 3$, $\omega_1 = 5$
- Third order $\phi_3(t,x) = \sum_{j=0}^{\infty} c_j(t)e_j(x)$, $(c_j, \dot c_j)|_{t=0} = 0$ and

$$\ddot c_j + \omega_j^2 c_j = \sum_k [\omega_k \in \Omega_3] b_{j,k} \cos(\omega_k t), \quad \text{where } \Omega_3 = \{|\omega_{0,1} \pm \omega_{0,1} \pm \omega_{0,1}|\}$$

Here $\Omega_3 = \{1,3,5,7,9,11,13,15\}$, but the the resonance ($b_{j,j} \neq 0$) only if $\omega_j \in \{3,5,7\}$.



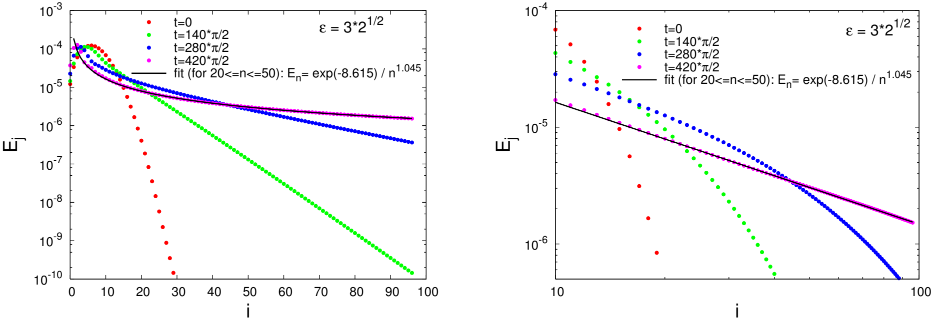
$\omega_0 \to \omega_0 + (87/\pi)\epsilon^2$, $\omega_1 \to \omega_1 + (413/\pi)\epsilon^2$ shifts remove the resonances $\omega_j = 3, 5$, but the resonance $\omega_j = 7$ cannot be removed. Thus we get the secular term $c_2(t) \sim t\sin(7t)$. We expect this term to be a progenitor of the onset of exponential instability.

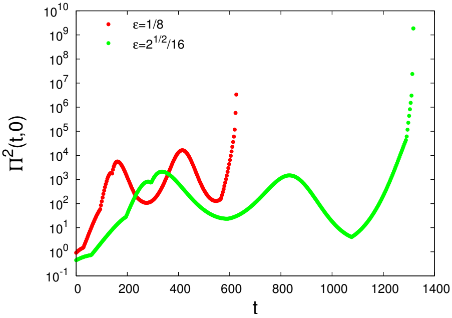
Turbulence: transfer of energy from low to high frequencies

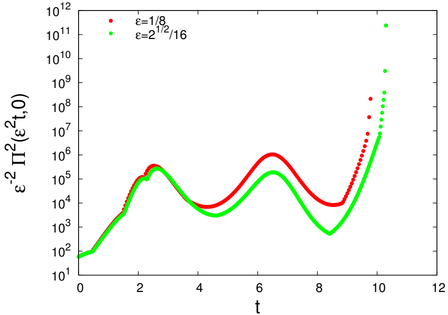
Let $\Pi_j := (\sqrt{A}\,\Pi, e_j)$ and $\Phi_j := (\sqrt{A}\,\Phi, e_j')$. Then

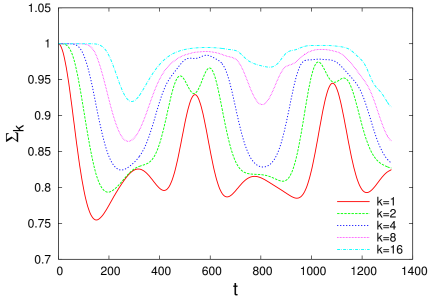
$$M = \frac{1}{2} \int_0^{\pi\ell/2} \left(A\Phi^2 + A\Pi^2\right) (\tan x)^2 \, dx = \sum_{j=0}^{\infty} E_j(t)\,,$$

where $E_j := \Pi_j^2 + \omega_j^{-2}\Phi_j^2$ can be interpreted as the j-mode energy.









$$\phi(0,x) = \varepsilon \left(e_0(x)/d_0 + e_1(x)/d_1 \right)$$

$$\Sigma_k := \frac{1}{M} \sum_{j=0}^{k} E_j$$

Conjectures

Our numerical and formal perturbative computations lead us to:

Conjecture 1

Anti-de Sitter space is unstable against the formation of a black hole under arbitrarily small generic perturbations

Proof (and a precise formulation) is left as a challenge. Note that we do not claim that all perturbed solutions end up as black holes.

Conjecture 2

There are non-generic initial data which may stay close to AdS solution; Einstein-scalar-AdS equations may admit time-quasiperiodic solutions

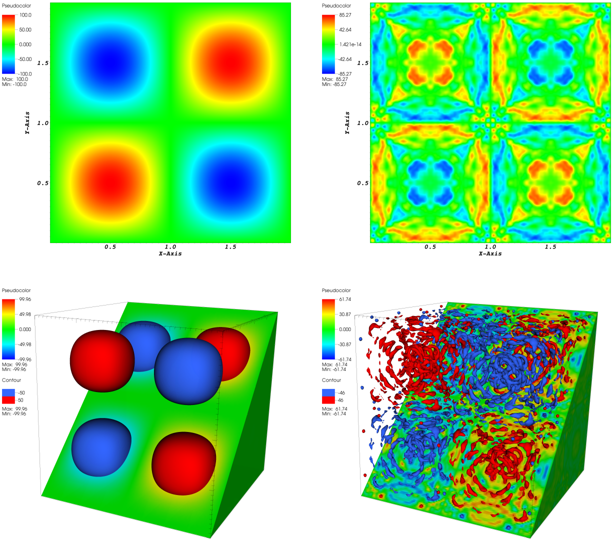
Proof: KAM theory for PDEs?

We hope that these conjectures will help to put rigorous studies of dynamics of asymptotically AdS spacetimes on the right track.

Other models and generalizations

- Qualitatively the same phenomenology for a self-gravitating scalar field in $d+1$ dimensions for any $d \geq 3$ (Jałmużna, Bizoń, R.)
- Work in progress by Jałmużna on $2+1$ AdS collapse of a scalar field. There is a mass gap for the formation of black holes (Pretorius&Choptuik 2000) $\Rightarrow$ for small perturbations turbulent instability cannot result in black hole formation
- Similar behavior for vacuum Einstein's equation in $4+1$ dimensions under the cohomogeneity-two biaxial Bianchi IX ansatz (Bizoń, R.)
- Related work in progress by Maliborski: Einstein-Yang-Mills AdS; extremely rich model (2 lengths scales, plenty of static solutions)
- Analogous weakly nonlinear perturbation analysis for $3+1$ vacuum Einstein's equations (Dias, Horowitz, Santos)

- Cubic defocusing nonlinear wave equation on a torus (Mach&Maliborski)



Final remarks

- Weakly turbulent behavior seems to be common for (non-integrable) nonlinear wave equations on bounded domains (e.g. NLS on torus, Colliander,Keel, Staffilani,Takaoka,Tao 2008, Carles,Faou 2010) and our work shows that Einstein's equations are not an exception.
- For Einstein's equations the transfer of energy to high frequencies cannot proceed forever because concentration of energy on smaller and smaller scales inevitably leads to the formation of a black hole.
- We believe that the role of negative cosmological constant is purely kinematical, that is the only role of Λ is to confine the evolution in an effectively bounded domain.