

# The PN Approximation beyond Point-Masses

Jan Steinhoff



Centro Multidisciplinar de Astrofísica (CENTRA)  
Instituto Superior Técnico (IST)

CENTRA Seminar, November 30th, 2011

**DFG** : STE 2017/1-1  
**FCT** : PTDC/CTEAST/098034/2008

# The post-Newtonian (PN) Approximation

of general relativity (GR)

- Newtonian limit: slow motion and weak field limit of GR
- PN approximation: expansion around Newtonian limit
- Bound binary: Virial theorem

$$2\langle \text{kinetic energy} \rangle_t = -\langle \text{potential energy} \rangle_t$$

$$2 \frac{1}{2} \mu v^2 \sim \frac{GM\mu}{r}$$

- one PN order  $\hat{=}$   $c^{-2}$
- half orders  $\hat{=}$   $c^{-1} \leftrightarrow$  antisymmetry under time-reversal  $\hat{=}$  radiation
- Assumption on  $T^{\mu\nu}$ : “strength” decreases as  $T^{00} \curvearrowright T^{i0} \curvearrowright T^{ij}$
- Source  $T^{\mu\nu}$  approximated by multipoles, parameter  $\sim \frac{R_{\text{object}}}{r}$

# The post-Newtonian (PN) Approximation

of general relativity (GR)

- Newtonian limit: slow motion and weak field limit of GR
- PN approximation: expansion around Newtonian limit
- Bound binary: Virial theorem

$$2\langle \text{kinetic energy} \rangle_t = -\langle \text{potential energy} \rangle_t$$

$$2 \frac{1}{2} \mu v^2 \sim \frac{GM\mu}{r}$$

- one PN order  $\hat{=}$   $c^{-2}$
- half orders  $\hat{=}$   $c^{-1} \leftrightarrow$  antisymmetry under time-reversal  $\hat{=}$  radiation
- Assumption on  $T^{\mu\nu}$ : “strength” decreases as  $T^{00} \curvearrowright T^{i0} \curvearrowright T^{ij}$
- Source  $T^{\mu\nu}$  approximated by multipoles, parameter  $\sim \frac{R_{\text{object}}}{r}$

# The post-Newtonian (PN) Approximation

of general relativity (GR)

- Newtonian limit: slow motion and weak field limit of GR
- PN approximation: expansion around Newtonian limit
- Bound binary: Virial theorem

$$2\langle \text{kinetic energy} \rangle_t = -\langle \text{potential energy} \rangle_t$$

$$\frac{v^2}{c^2} \sim \frac{GM}{c^2 r}$$

- one PN order  $\hat{=}$   $c^{-2}$
- half orders  $\hat{=}$   $c^{-1} \leftrightarrow$  antisymmetry under time-reversal  $\hat{=}$  radiation
- Assumption on  $T^{\mu\nu}$ : “strength” decreases as  $T^{00} \curvearrowright T^{i0} \curvearrowright T^{ij}$
- Source  $T^{\mu\nu}$  approximated by multipoles, parameter  $\sim \frac{R_{\text{object}}}{r}$

# The post-Newtonian (PN) Approximation

of general relativity (GR)

- Newtonian limit: slow motion and weak field limit of GR
- PN approximation: expansion around Newtonian limit
- Bound binary: Virial theorem

$$2\langle \text{kinetic energy} \rangle_t = -\langle \text{potential energy} \rangle_t$$

$$\frac{v^2}{c^2} \sim \frac{GM}{c^2 r} \sim \begin{array}{l} \text{dimensionless} \\ \text{expansion parameter} \end{array}$$

- one PN order  $\hat{=}$   $c^{-2}$
- half orders  $\hat{=}$   $c^{-1} \leftrightarrow$  antisymmetry under time-reversal  $\hat{=}$  radiation
- Assumption on  $T^{\mu\nu}$ : “strength” decreases as  $T^{00} \curvearrowright T^{i0} \curvearrowright T^{ij}$
- Source  $T^{\mu\nu}$  approximated by multipoles, parameter  $\sim \frac{R_{\text{object}}}{r}$

# The post-Newtonian (PN) Approximation

of general relativity (GR)

- Newtonian limit: slow motion and weak field limit of GR
- PN approximation: expansion around Newtonian limit
- Bound binary: Virial theorem

$$2\langle \text{kinetic energy} \rangle_t = -\langle \text{potential energy} \rangle_t$$

$$\frac{v^2}{c^2} \sim \frac{GM}{c^2 r} \sim \text{dimensionless expansion parameter}$$

- one PN order  $\hat{=}$   $c^{-2}$
- half orders  $\hat{=}$   $c^{-1}$   $\leftrightarrow$  antisymmetry under time-reversal  $\hat{=}$  radiation
- Assumption on  $T^{\mu\nu}$ : “strength” decreases as  $T^{00} \curvearrowright T^{i0} \curvearrowright T^{ij}$
- Source  $T^{\mu\nu}$  approximated by multipoles, parameter  $\sim \frac{R_{\text{object}}}{r}$

# The post-Newtonian (PN) Approximation

of general relativity (GR)

- Newtonian limit: slow motion and weak field limit of GR
- PN approximation: expansion around Newtonian limit
- Bound binary: Virial theorem

$$2\langle \text{kinetic energy} \rangle_t = -\langle \text{potential energy} \rangle_t$$

$$\frac{v^2}{c^2} \sim \frac{GM}{c^2 r} \sim \begin{array}{l} \text{dimensionless} \\ \text{expansion parameter} \end{array}$$

- one PN order  $\hat{=}$   $c^{-2}$
- half orders  $\hat{=}$   $c^{-1}$   $\leftrightarrow$  antisymmetry under time-reversal  $\hat{=}$  radiation
- Assumption on  $T^{\mu\nu}$ : “strength” decreases as  $T^{00} \curvearrowright T^{i0} \curvearrowright T^{ij}$
- Source  $T^{\mu\nu}$  approximated by multipoles, parameter  $\sim \frac{R_{\text{object}}}{r}$

# Electrostatic Multipoles

- Taylor series of Fourier-transformed charge density  $\rho$ :

$$\rho(\mathbf{k}) = \left( q + iq^i k_i + \frac{1}{2!} i^2 q^{ij} k_i k_j + \dots \right) (2\pi)^{-3/2}$$

- In position space:

$$\rho(\mathbf{x}) = \left( q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \delta(\mathbf{x})$$

$$\delta(\mathbf{x}) \leftrightarrow (2\pi)^{-3/2} \quad \partial_i \leftrightarrow -ik_i$$

- The potential  $\phi$  reads:

$$\phi = -4\pi \Delta^{-1} \rho = \left( q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \frac{1}{|\mathbf{x}|}$$

- Electric monopole, dipole, and quadrupole:  $q, q^i, q^{ij}$
- Multipole approximation breaks down for big  $|\mathbf{k}|$  or small  $|\mathbf{x}|$  ( $\sim R_{\text{object}}$ )
- Self-energy UV-divergent:  $\int \rho \phi \sim \frac{1}{0}$



# Electrostatic Multipoles

- Taylor series of Fourier-transformed charge density  $\rho$ :

$$\rho(\mathbf{k}) = \left( q + iq^i k_i + \frac{1}{2!} i^2 q^{ij} k_i k_j + \dots \right) (2\pi)^{-3/2}$$

- In position space:

$$\rho(\mathbf{x}) = \left( q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \delta(\mathbf{x})$$

$$\delta(\mathbf{x}) \leftrightarrow (2\pi)^{-3/2} \quad \partial_i \leftrightarrow -ik_i$$

- The potential  $\phi$  reads:

$$\phi = -4\pi \Delta^{-1} \rho = \left( q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \frac{1}{|\mathbf{x}|}$$

- Electric monopole, dipole, and quadrupole:  $q, q^i, q^{ij}$
- Multipole approximation breaks down for big  $|\mathbf{k}|$  or small  $|\mathbf{x}|$  ( $\sim R_{\text{object}}$ )
- Self-energy UV-divergent:  $\int \rho \phi \sim \frac{1}{0}$

# Electrostatic Multipoles

- Taylor series of Fourier-transformed charge density  $\rho$ :

$$\rho(\mathbf{k}) = \left( q + iq^i k_i + \frac{1}{2!} i^2 q^{ij} k_i k_j + \dots \right) (2\pi)^{-3/2}$$

- In position space:

$$\rho(\mathbf{x}) = \left( q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \delta(\mathbf{x})$$

$$\delta(\mathbf{x}) \leftrightarrow (2\pi)^{-3/2} \quad \partial_i \leftrightarrow -ik_i$$

- The potential  $\phi$  reads:

$$\phi = -4\pi \Delta^{-1} \rho = \left( q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \frac{1}{|\mathbf{x}|}$$

- Electric monopole, dipole, and quadrupole:  $q, q^i, q^{ij}$
- Multipole approximation breaks down for big  $|\mathbf{k}|$  or small  $|\mathbf{x}|$  ( $\sim R_{\text{object}}$ )
- Self-energy UV-divergent:  $\int \rho \phi \sim \frac{1}{0}$

# Electrostatic Multipoles

- Taylor series of Fourier-transformed charge density  $\rho$ :

$$\rho(\mathbf{k}) = \left( q + iq^i k_i + \frac{1}{2!} i^2 q^{ij} k_i k_j + \dots \right) (2\pi)^{-3/2}$$

- In position space:

$$\rho(\mathbf{x}) = \left( q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \delta(\mathbf{x})$$

$$\delta(\mathbf{x}) \leftrightarrow (2\pi)^{-3/2} \quad \partial_i \leftrightarrow -ik_i$$

- The potential  $\phi$  reads:

$$\phi = -4\pi \Delta^{-1} \rho = \left( q - q^i \partial_i + \frac{1}{2!} q^{ij} \partial_i \partial_j - \dots \right) \frac{1}{|\mathbf{x}|}$$

- Electric monopole, dipole, and quadrupole:  $q$ ,  $q^i$ ,  $q^{ij}$
- Multipole approximation breaks down for big  $|\mathbf{k}|$  or small  $|\mathbf{x}|$  ( $\sim R_{\text{object}}$ )
- Self-energy UV-divergent:  $\int \rho \phi \sim \frac{1}{0}$

# Gravitational Multipoles

and the PN approximation beyond point-masses

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \left[ u^{(\mu} p^{\nu)} \delta_{(4)} + \left( u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} \right. \\ \left. + \frac{1}{3} R_{\alpha\beta\rho}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left( J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} + \dots \right]$$

$$u^\mu = \frac{dz^\mu}{d\tau} \quad \delta_{(4)} = \delta(x^\sigma - z^\sigma)$$

- Point masses only distinguished by a mass  $m = \sqrt{-p_\mu p^\mu}$
- Adding a dipole: Spin
- Higher multipoles: Quadrupole, octupole, ... (“finite size effects”)
- Dimensional regularization required for self-gravitating objects

# Gravitational Multipoles

and the PN approximation beyond point-masses

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \left[ u^{(\mu} p^{\nu)} \delta_{(4)} + \left( u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} + \frac{1}{3} R_{\alpha\beta\rho}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left( J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} + \dots \right]$$

$$u^\mu = \frac{dz^\mu}{d\tau} \quad \delta_{(4)} = \delta(x^\sigma - z^\sigma)$$

- Point masses only distinguished by a mass  $m = \sqrt{-p_\mu p^\mu}$
- Adding a dipole: Spin
- Higher multipoles: Quadrupole, octupole, ... (“finite size effects”)
- Dimensional regularization required for self-gravitating objects

# Gravitational Multipoles

and the PN approximation beyond point-masses

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \left[ u^{(\mu} p^{\nu)} \delta_{(4)} + \left( u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} \right. \\ \left. + \frac{1}{3} R_{\alpha\beta\rho}{}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left( J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} + \dots \right]$$

$$u^\mu = \frac{dz^\mu}{d\tau} \quad \delta_{(4)} = \delta(x^\sigma - z^\sigma)$$

- Point masses only distinguished by a mass  $m = \sqrt{-p_\mu p^\mu}$
- Adding a dipole: **Spin**
- Higher multipoles: **Quadrupole**, octupole, ... (“finite size effects”)
- Dimensional regularization required for self-gravitating objects

# Gravitational Multipoles

and the PN approximation beyond point-masses

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \left[ u^{(\mu} p^{\nu)} \delta_{(4)} + \left( u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} \right. \\ \left. + \frac{1}{3} R_{\alpha\beta\rho}{}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left( J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} + \dots \right]$$

$$u^\mu = \frac{dz^\mu}{d\tau} \quad \delta_{(4)} = \delta(x^\sigma - z^\sigma)$$

- Point masses only distinguished by a mass  $m = \sqrt{-p_\mu p^\mu}$
- Adding a dipole: **Spin**
- Higher multipoles: **Quadrupole**, octupole, ... (“finite size effects”)
- Dimensional regularization required for self-gravitating objects

# Gravitational Multipoles

and the PN approximation beyond point-masses

$$\sqrt{-g}T^{\mu\nu}(x^\sigma) = \int d\tau \left[ u^{(\mu} p^{\nu)} \delta_{(4)} + \left( u^{(\mu} S^{\nu)\alpha} \delta_{(4)} \right)_{||\alpha} \right. \\ \left. + \frac{1}{3} R_{\alpha\beta\rho}^{(\mu} J^{\nu)\rho\beta\alpha} \delta_{(4)} - \frac{2}{3} \left( J^{\mu\alpha\beta\nu} \delta_{(4)} \right)_{||(\alpha\beta)} + \dots \right]$$

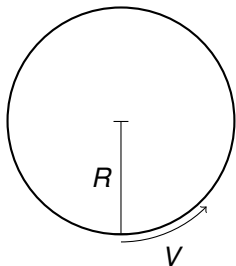
$$u^\mu = \frac{dz^\mu}{d\tau} \quad \delta_{(4)} = \delta(x^\sigma - z^\sigma)$$

- Point masses only distinguished by a mass  $m = \sqrt{-p_\mu p^\mu}$
- Adding a dipole: **Spin**
- Higher multipoles: **Quadrupole**, octupole, ... (“finite size effects”)
- Dimensional regularization required for self-gravitating objects



# Two Facts on Spin in Relativity

## 1. Minimal Extension



- ring of radius  $R$  and mass  $M$
- spin:  $S = R M V$
- maximal velocity:  $V \leq c$   
 $\Rightarrow$  minimal extension:

$$R = \frac{S}{M V} \geq \frac{S}{M c}$$

## 2. Center-of-mass

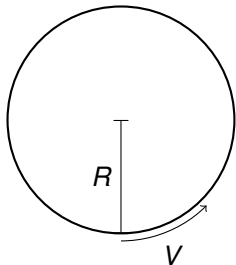


- now moving with velocity  $v$
- relativistic mass changes inhom.
- frame-dependent center-of-mass
- need spin supplementary condition,

e.g.,  $S^{\mu\nu} p_\nu = 0$

# Two Facts on Spin in Relativity

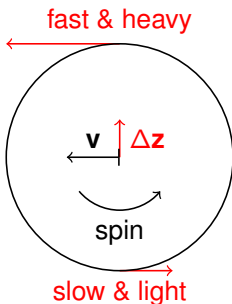
## 1. Minimal Extension



- ring of radius  $R$  and mass  $M$
- spin:  $S = R M V$
- maximal velocity:  $V \leq c$   
 $\Rightarrow$  minimal extension:

$$R = \frac{S}{MV} \geq \frac{S}{Mc}$$

## 2. Center-of-mass

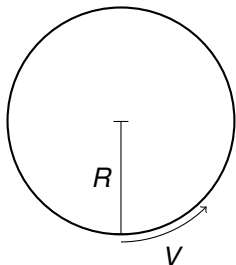


- now moving with velocity  $v$
- relativistic mass changes inhom.
- frame-dependent center-of-mass
- need spin supplementary condition,

e.g.,  $S^{\mu\nu} p_\nu = 0$

# Two Facts on Spin in Relativity

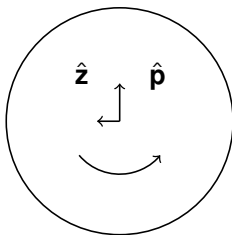
## 1. Minimal Extension



- ring of radius  $R$  and mass  $M$
- spin:  $S = R M V$
- maximal velocity:  $V \leq c$   
 $\Rightarrow$  minimal extension:

$$R = \frac{S}{MV} \geq \frac{S}{Mc}$$

## 2. Center-of-mass



- now moving with velocity  $v$
- relativistic mass changes inhom.
- frame-dependent center-of-mass
- need spin supplementary condition,

e.g.,  $S^{\mu\nu} p_\nu = 0$

# Angular Velocity and Spin

in Newtonian mechanics and special relativity

	Newton	special relativity
body-fixed frame	$x_{\text{bf}}^i = \Lambda^{ij} x^j$	
rotational degrees of freedom ↪ supplementary condition	$\Lambda^{ki} \Lambda^{kj} = \delta_{ij}$	$\eta_{AB} \Lambda^{A\mu} \Lambda^{B\nu} = \eta^{\mu\nu}$ $\Lambda_{i\mu} p^\mu = 0$
Angular Velocity	$\Omega^{ij} = \Lambda^{ki} \frac{d\Lambda^{kj}}{dt}$	$\Omega^{\mu\nu} = \Lambda_A^\mu \frac{d\Lambda^{A\nu}}{d\tau}$
Spin ( $L$ : Lagrangian) ↪ supplementary condition	$S_{ij} = 2 \frac{\partial L}{\partial \Omega^{ij}}$	$S_{\mu\nu} = 2 \frac{\partial L}{\partial \Omega^{\mu\nu}}$ $S_{\mu\nu} p^\nu = 0$

Remark:

- Angular velocity vector is  $\Omega^i = \frac{1}{2} \epsilon_{ijk} \Omega^{jk}$ . Analogous for spin.

# Angular Velocity and Spin

in Newtonian mechanics and special relativity

	Newton	special relativity
body-fixed frame	$x_{\text{bf}}^i = \Lambda^{ij} x^j$	
rotational degrees of freedom ↪ supplementary condition	$\Lambda^{ki} \Lambda^{kj} = \delta_{ij}$	$\eta_{AB} \Lambda^{A\mu} \Lambda^{B\nu} = \eta^{\mu\nu}$ $\Lambda_{i\mu} p^\mu = 0$
Angular Velocity	$\Omega^{ij} = \Lambda^{ki} \frac{d\Lambda^{kj}}{dt}$	$\Omega^{\mu\nu} = \Lambda_A^\mu \frac{d\Lambda^{A\nu}}{d\tau}$
Spin ( $L$ : Lagrangian) ↪ supplementary condition	$S_{ij} = 2 \frac{\partial L}{\partial \Omega^{ij}}$	$S_{\mu\nu} = 2 \frac{\partial L}{\partial \Omega^{\mu\nu}}$ $S_{\mu\nu} p^\nu = 0$

Remark:

- Angular velocity vector is  $\Omega^i = \frac{1}{2} \epsilon_{ijk} \Omega^{jk}$ . Analogous for spin.

# Spin Action in GR

- Minimal coupling:

$$\Omega^{\mu\nu} = \Lambda_A{}^\mu \frac{D\Lambda^{A\nu}}{d\tau}$$

$$L = m_c \underbrace{\sqrt{-u_\mu u^\mu}}_u + \underbrace{\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu}}_{\sim \frac{1}{2} S^{ij} \partial_i A_j} + \dots$$

- $m \approx m_c = \text{const}$
- Valid to linear order in spin
- Gravito-magnetic field  $A_i \approx -g_{i0}$
- Metric variation problematic:

$$\Lambda_{A\mu} \Lambda^A{}_\nu = g_{\mu\nu} \quad \leftrightarrow \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$$

- Variate  $\Lambda^{Aa}$  and tetrad  $e_{a\mu}$ :  $e_{a\mu} e^a{}_\nu = g_{\mu\nu} \quad \Lambda^A{}_\mu = \Lambda^{Aa} e_{a\mu}$

$$\Lambda_{Aa} \Lambda^A{}^b = \eta_{ab} \quad \leftrightarrow \quad \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}$$

# Spin Action in GR

- Minimal coupling:

$$\Omega^{\mu\nu} = \Lambda_A{}^\mu \frac{D\Lambda^{A\nu}}{d\tau}$$

$$L = m_c \underbrace{\sqrt{-u_\mu u^\mu}}_u + \underbrace{\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu}}_{\sim \frac{1}{2} S^i{}_j \partial_i A_j} + \dots$$

- $m \approx m_c = \text{const}$
- Valid to linear order in spin
- Gravito-magnetic field  $A_i \approx -g_{i0}$
- Metric variation problematic:

$$\Lambda_{A\mu} \Lambda^A{}_\nu = g_{\mu\nu} \quad \leftrightarrow \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$$

- Variate  $\Lambda^{Aa}$  and tetrad  $e_{a\mu}$ :  $e_{a\mu} e^a{}_\nu = g_{\mu\nu} \quad \Lambda^A{}_\mu = \Lambda^{Aa} e_{a\mu}$

$$\Lambda_{Aa} \Lambda^A{}_b = \eta_{ab} \quad \leftrightarrow \quad \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}$$

# Spin Action in GR

- Minimal coupling:

$$\Omega^{\mu\nu} = \Lambda_A{}^\mu \frac{D\Lambda^{A\nu}}{d\tau}$$

$$L = m_c \underbrace{\sqrt{-u_\mu u^\mu}}_u + \underbrace{\frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu}}_{\sim \frac{1}{2} S^{ij} \partial_i A_j} + \dots$$

- $m \approx m_c = \text{const}$
- Valid to linear order in spin
- Gravito-magnetic field  $A_i \approx -g_{i0}$
- Metric variation problematic:

$$\Lambda_{A\mu} \Lambda^A{}_\nu = g_{\mu\nu} \quad \leftrightarrow \quad \gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}$$

- Variate  $\Lambda^{Aa}$  and tetrad  $e_{a\mu}$ :  $e_{a\mu} e^a{}_\nu = g_{\mu\nu} \quad \Lambda^A{}_\mu = \Lambda^{Aa} e_{a\mu}$

$$\Lambda_{Aa} \Lambda^A{}_b = \eta_{ab} \quad \leftrightarrow \quad \gamma_a \gamma_b + \gamma_b \gamma_a = 2\eta_{ab}$$



# Spin and Gravitomagnetism



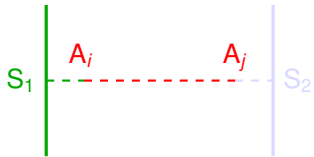
$$\begin{aligned}
 & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad 16\pi G \delta_{ij} \Delta^{-1} \quad \frac{1}{2} S_2^{lj} \partial_l \delta_2 \\
 = & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad (-2) G S_2^{lj} \partial_l \left( \frac{1}{r_2} \right) \\
 = & G S_1^{ki} S_2^{lj} \partial_k \partial_l \left( \frac{1}{r_1} \right) \Big|_{\mathbf{x}=\mathbf{z}_1}
 \end{aligned}$$

- We will calculate the leading-order  $S_1 S_2$  potential
- Here:  $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$ ,  $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of  $T^{00}$ ,  $T^{i0}$ ,  $T^{ij}$  revised

N	mass $T^{00}$	$\rightsquigarrow$ gravito-electric field
1PN	flow $T^{i0}$	$\rightsquigarrow$ gravito-magnetic field ( $A_i$ )
2PN	stress $T^{ij}$	$\rightsquigarrow$ 3-dim. tensor field

# Spin and Gravitomagnetism



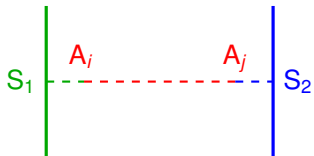
$$\begin{aligned}
 & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad 16\pi G \delta_{ij} \Delta^{-1} \quad \frac{1}{2} S_2^{lj} \partial_l \delta_2 \\
 = & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad (-2) G S_2^{lj} \partial_l \left( \frac{1}{r_2} \right) \\
 = & G S_1^{ki} S_2^{lj} \partial_k \partial_l \left( \frac{1}{r_1} \right) \Big|_{\mathbf{x}=\mathbf{z}_1}
 \end{aligned}$$

- We will calculate the leading-order  $S_1 S_2$  potential
- Here:  $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$ ,  $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of  $T^{00}$ ,  $T^{i0}$ ,  $T^{ij}$  revised

N	mass $T^{00}$	$\rightsquigarrow$ gravito-electric field
1PN	flow $T^{i0}$	$\rightsquigarrow$ gravito-magnetic field ( $A_i$ )
2PN	stress $T^{ij}$	$\rightsquigarrow$ 3-dim. tensor field

# Spin and Gravitomagnetism



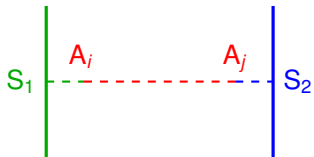
$$\begin{aligned}
 & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad 16\pi G \delta_{ij} \Delta^{-1} \quad \frac{1}{2} S_2^{lj} \partial_l \delta_2 \\
 &= \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 (-2) G S_2^{lj} \partial_l \left( \frac{1}{r_2} \right) \\
 &= G S_1^{ki} S_2^{lj} \partial_k \partial_l \left( \frac{1}{r_1} \right) \Big|_{\mathbf{x}=\mathbf{z}_1}
 \end{aligned}$$

- We will calculate the leading-order  $S_1 S_2$  potential
- Here:  $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$ ,  $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of  $T^{00}$ ,  $T^{i0}$ ,  $T^{ij}$  revised

N	mass $T^{00}$	$\rightsquigarrow$ gravito-electric field
1PN	flow $T^{i0}$	$\rightsquigarrow$ gravito-magnetic field ( $A_i$ )
2PN	stress $T^{ij}$	$\rightsquigarrow$ 3-dim. tensor field

# Spin and Gravitomagnetism



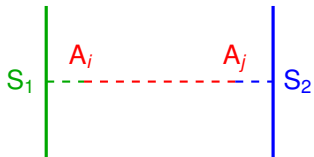
$$\begin{aligned}
 & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad 16\pi G \delta_{ij} \Delta^{-1} \quad \frac{1}{2} S_2^{lj} \partial_l \delta_2 \\
 &= \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 (-2) G S_2^{lj} \partial_l \left( \frac{1}{r_2} \right) \\
 &= G S_1^{ki} S_2^{lj} \partial_k \partial_l \left( \frac{1}{r_1} \right) \Big|_{\mathbf{x}=\mathbf{z}_1}
 \end{aligned}$$

- We will calculate the leading-order  $S_1 S_2$  potential
- Here:  $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$ ,  $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of  $T^{00}$ ,  $T^{i0}$ ,  $T^{ij}$  revised

N	mass $T^{00}$	$\rightsquigarrow$ gravito-electric field
1PN	flow $T^{i0}$	$\rightsquigarrow$ gravito-magnetic field ( $A_i$ )
2PN	stress $T^{ij}$	$\rightsquigarrow$ 3-dim. tensor field

# Spin and Gravitomagnetism



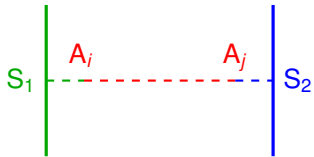
$$\begin{aligned}
 & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad 16\pi G \delta_{ij} \Delta^{-1} \quad \frac{1}{2} S_2^{lj} \partial_l \delta_2 \\
 = & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad (-2) G S_2^{lj} \partial_l \left( \frac{1}{r_2} \right) \\
 = & G S_1^{ki} S_2^{lj} \partial_k \partial_l \left( \frac{1}{r_1} \right) \Big|_{\mathbf{x}=\mathbf{z}_1}
 \end{aligned}$$

- We will calculate the leading-order  $S_1 S_2$  potential
- Here:  $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$ ,  $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of  $T^{00}$ ,  $T^{i0}$ ,  $T^{ij}$  revised

N	mass $T^{00}$	$\rightsquigarrow$ gravito-electric field
1PN	flow $T^{i0}$	$\rightsquigarrow$ gravito-magnetic field ( $A_i$ )
2PN	stress $T^{ij}$	$\rightsquigarrow$ 3-dim. tensor field

# Spin and Gravitomagnetism



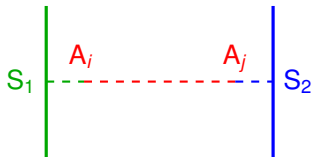
$$\begin{aligned}
 & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad 16\pi G \delta_{ij} \Delta^{-1} \quad \frac{1}{2} S_2^{lj} \partial_l \delta_2 \\
 = & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 (-2) G S_2^{lj} \partial_l \left( \frac{1}{r_2} \right) \\
 = & G S_1^{ki} S_2^{lj} \partial_k \partial_l \left( \frac{1}{r_1} \right) \Big|_{\mathbf{x}=\mathbf{z}_1}
 \end{aligned}$$

- We will calculate the leading-order  $S_1 S_2$  potential
- Here:  $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$ ,  $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of  $T^{00}$ ,  $T^{i0}$ ,  $T^{ij}$  revised

N	mass $T^{00}$	$\rightsquigarrow$ gravito-electric field
1PN	flow $T^{i0}$	$\rightsquigarrow$ gravito-magnetic field ( $A_i$ )
2PN	stress $T^{ij}$	$\rightsquigarrow$ 3-dim. tensor field

# Spin and Gravitomagnetism



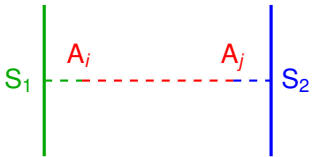
$$\begin{aligned}
 & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad 16\pi G \delta_{ij} \Delta^{-1} \quad \frac{1}{2} S_2^{lj} \partial_l \delta_2 \\
 = & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 (-2) G S_2^{lj} \partial_l \left( \frac{1}{r_2} \right) \\
 = & G S_1^{ki} S_2^{lj} \partial_k \partial_l \left( \frac{1}{r_1} \right) \Big|_{\mathbf{x}=\mathbf{z}_1}
 \end{aligned}$$

- We will calculate the leading-order  $S_1 S_2$  potential
- Here:  $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$ ,  $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

Relevance of  $T^{00}$ ,  $T^{i0}$ ,  $T^{ij}$  revised

N	mass $T^{00}$	$\rightsquigarrow$ gravito-electric field
1PN	flow $T^{i0}$	$\rightsquigarrow$ gravito-magnetic field ( $A_i$ )
2PN	stress $T^{ij}$	$\rightsquigarrow$ 3-dim tensor field

# Spin and Gravitomagnetism



$$\begin{aligned}
 & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad 16\pi G \delta_{ij} \Delta^{-1} \quad \frac{1}{2} S_2^{lj} \partial_l \delta_2 \\
 = & \int d^3x \frac{1}{2} S_1^{ki} \partial_k \delta_1 \quad (-2) G S_2^{lj} \partial_l \left( \frac{1}{r_2} \right) \\
 = & G S_1^{ki} S_2^{lj} \partial_k \partial_l \left( \frac{1}{r_1} \right) \Big|_{\mathbf{x}=\mathbf{z}_1}
 \end{aligned}$$

- We will calculate the leading-order  $S_1 S_2$  potential
- Here:  $\delta_a = \delta(\mathbf{x} - \mathbf{z}_a)$ ,  $r_a = |\mathbf{x} - \mathbf{z}_a|$
- Diagrams encode integrals
- Translation rules: Feynman rules see e.g. Levi arXiv:1006.4139
- Rules follow from the action

## Relevance of $T^{00}$ , $T^{i0}$ , $T^{ij}$ revised

N	mass $T^{00}$	$\rightsquigarrow$ gravito-electric field
1PN	flow $T^{i0}$	$\rightsquigarrow$ gravito-magnetic field ( $A_i$ )
2PN	stress $T^{ij}$	$\rightsquigarrow$ 3-dim. tensor field



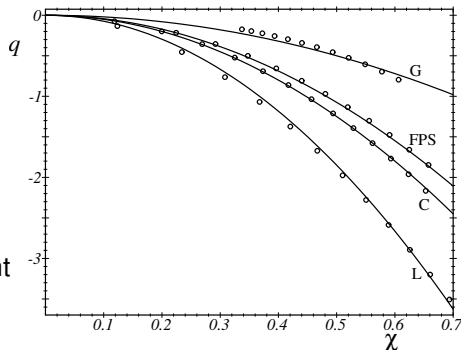
# Quadrupole Deformation due to Spin

for neutron stars, Laarakkers, Poisson gr-qc/9709033

- Here  $m = 1.4M_{\odot}$
- Dim.-less mass quadrupole:  $q$
- Dim.-less spin:  $\chi$
- Quadratic fit is extremely good:

$$-q \approx C_{ES^2} \chi^2$$

- $C_{ES^2} = 4.3 \dots 7.4$ , EOS dependent
- Also depends on mass



plot from Laarakkers, Poisson gr-qc/9709033

- For black holes  $C_{ES^2} = 1$
- $S^4$ -quadrupole is highly suppressed
- RNS code by N. Stergioulas publicly available

# Tidal Quadrupole Deformation

for neutron stars, e.g. Damour, Nagar arXiv:0906.0096, Binnington, Poisson arXiv:0906.1366

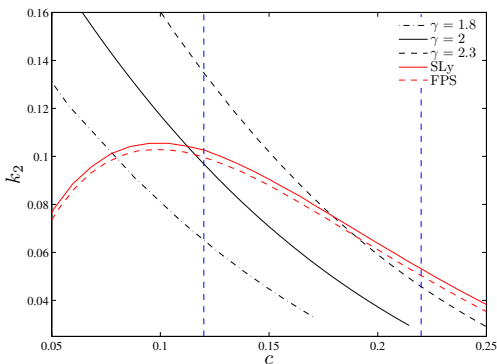
- Linear NS perturbation, thus:

$$-Q = \mu_2 E$$

- Tidal force  $E$  (curvature)
- Dim.-less 2nd Love number  $k_2$ :

$$k_2 = \frac{3}{2} \frac{\mu_2}{R^5}$$

- Compactness  $c = \frac{Gm}{R}$



plot from Damour, Nagar arXiv:0906.0096

- For certain realistic EOS it holds  $k_2 \approx 0.17 - 0.52c$
- For black holes  $k_2 \sim 0$

# Quadrupole Action

see e.g. Porto, Rothstein arXiv:0804.0260, Goldberger, Rothstein hep-th/0511133

$$L_{\text{quad}} = \underbrace{\frac{1}{m_c u} B_{\mu\nu} S^\mu u_\alpha S^{\alpha\nu}}_{\text{SSC preserving}} + \underbrace{\frac{C_{ES^2}}{2m_c u} E_{\mu\nu} S^\mu{}_\alpha S^{\alpha\nu}}_{\text{deformation due to spin}} + \underbrace{\frac{\mu_2}{4u^3} E_{\mu\nu} E^{\mu\nu}}_{\text{tidal deformation}} + \dots$$

$$E_{\mu\nu} \sim R_{\mu\alpha\nu\beta} u^\alpha u^\beta \quad B_{\mu\nu} \sim \frac{1}{2} \epsilon_{\mu\rho\alpha\beta} R_{\nu\sigma}{}^{\alpha\beta} u^\rho u^\sigma \quad S^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu S_{\alpha\beta}$$

- $m_c$ ,  $C_{ES^2}$ , and  $\mu_2$ : constants, matched to single object
- Now:  $m_c \neq m$
- From Bailey, Israel (1975):

$$J^{\mu\nu\alpha\beta} = -6 \frac{\partial L}{\partial R_{\mu\nu\alpha\beta}}$$

- Covariant mass quadrupole: (for  $u = 1$ )

$$\text{mass quadrupole} \sim 2 \frac{\partial L}{\partial E_{\mu\nu}} = \frac{C_{ES^2}}{m_c} S^\mu{}_\alpha S^{\alpha\nu} + \mu_2 E^{\mu\nu}$$

# Surface Terms and ADM Hamiltonian

ADM  $\triangleq$  Arnowitt, Deser, Misner

- Einstein–Hilbert action plus York–Gibbons–Hawking surface term:

$$S_{\text{field}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{16\pi G} \oint d^3y 2\sqrt{h} K$$

- ADM energy given by surface integral

$$E_{\text{ADM}} = \frac{1}{16\pi G} \oint d^2s_i [\gamma_{ij,j} - \gamma_{j,i}]$$

- $H^{\text{ADM}} \triangleq$  ADM energy  $E_{\text{ADM}}$  expressed in terms of canonical variables
- Canonical field variables:  $h_{ij}^{\text{TT}}, \pi^{ij\text{TT}}$       TT  $\triangleq$  transverse-traceless

DeWitt (1967)

“General relativity is unique among field theories in that its energy may always be expressed as a surface integral.”

# Surface Terms and ADM Hamiltonian

ADM  $\hat{=}$  Arnowitt, Deser, Misner

- Einstein–Hilbert action plus York–Gibbons–Hawking surface term:

$$S_{\text{field}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{16\pi G} \oint d^3y 2\sqrt{h} K$$

- ADM energy given by surface integral

$$E_{\text{ADM}} = \frac{1}{16\pi G} \oint d^2s_i [\gamma_{ij,j} - \gamma_{jj,i}]$$

- $H^{\text{ADM}} \hat{=}$  ADM energy  $E_{\text{ADM}}$  expressed in terms of canonical variables
- Canonical field variables:  $h_{ij}^{\text{TT}}, \pi^{ij\text{TT}}$       TT  $\hat{=}$  transverse-traceless

DeWitt (1967)

“General relativity is unique among field theories in that its energy may always be expressed as a surface integral.”

# Surface Terms and ADM Hamiltonian

ADM  $\triangleq$  Arnowitt, Deser, Misner

- Einstein–Hilbert action plus York–Gibbons–Hawking surface term:

$$S_{\text{field}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{16\pi G} \oint d^3y 2\sqrt{h} K$$

- ADM energy given by surface integral

$$E_{\text{ADM}} = \frac{1}{16\pi G} \oint d^2s_i [\gamma_{ij,j} - \gamma_{jj,i}]$$

- $H^{\text{ADM}} \triangleq$  ADM energy  $E_{\text{ADM}}$  expressed in terms of canonical variables
- Canonical field variables:  $h_{ij}^{\text{TT}}, \pi^{ij\text{TT}}$       TT  $\triangleq$  transverse-traceless

DeWitt (1967)

“General relativity is unique among field theories in that its energy may always be expressed as a surface integral.”

# Surface Terms and ADM Hamiltonian

ADM  $\triangleq$  Arnowitt, Deser, Misner

- Einstein–Hilbert action plus York–Gibbons–Hawking surface term:

$$S_{\text{field}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \frac{1}{16\pi G} \oint d^3y 2\sqrt{h} K$$

- ADM energy given by surface integral

$$E_{\text{ADM}} = \frac{1}{16\pi G} \oint d^2s_i [\gamma_{ij,j} - \gamma_{jj,i}]$$

- $H^{\text{ADM}} \triangleq$  ADM energy  $E_{\text{ADM}}$  expressed in terms of canonical variables
- Canonical field variables:  $h_{ij}^{\text{TT}}, \pi^{ij\text{TT}}$       TT  $\triangleq$  transverse-traceless

DeWitt (1967)

“General relativity is unique among field theories in that its energy may always be expressed as a surface integral.”

# Canonical Variables to Linear Order in Spin

- Flat spacetime: Newton-Wigner center and spin are canonical
- Canonical  $\hat{Z}^i$ ,  $\hat{S}_{ij}$ , and  $\hat{\Lambda}^{ij}$  are “simple” generalizations of flat space case
- Canonical matter momentum  $\hat{p}_i$ :

$$p_i = \hat{p}_i + \frac{1}{2} \hat{S}_{kj} \Gamma^{kj}_i + \dots$$

cf. electrodynamics:  $p_i = \hat{p}_i - qA_i$

- Canonical field momentum  $\hat{\pi}^{ij\text{TT}}$  has **delta-corrections**:

$$\pi^{ij\text{TT}} = \hat{\pi}^{ij\text{TT}} + \frac{4\pi G}{m^2} \hat{p}_m \hat{p}_k \hat{S}^{lm} \delta_{kl}^{\text{TT}ij} \delta + \dots$$

can not be given in closed form explicitly



# Canonical Variables to Linear Order in Spin

- Flat spacetime: Newton-Wigner center and spin are canonical
- Canonical  $\hat{Z}^i$ ,  $\hat{S}_{ij}$ , and  $\hat{\Lambda}^{ij}$  are “simple” generalizations of flat space case
- Canonical matter momentum  $\hat{p}_i$ :

$$p_i = \hat{p}_i + \frac{1}{2} \hat{S}_{kj} \Gamma_{i}^{kj} + \dots$$

cf. electrodynamics:  $p_i = \hat{p}_i - qA_i$

- Canonical field momentum  $\hat{\pi}^{ij\text{TT}}$  has **delta-corrections**:

$$\pi^{ij\text{TT}} = \hat{\pi}^{ij\text{TT}} + \frac{4\pi G}{m^2} \hat{p}_m \hat{p}_k \hat{S}^{lm} \delta_{kl}^{\text{TT}ij} \delta + \dots$$

can not be given in closed form explicitly

# Canonical Variables to Linear Order in Spin

- Flat spacetime: Newton-Wigner center and spin are canonical
- Canonical  $\hat{Z}^i$ ,  $\hat{S}_{ij}$ , and  $\hat{\Lambda}^{ij}$  are “simple” generalizations of flat space case
- Canonical matter momentum  $\hat{p}_i$ :

$$p_i = \hat{p}_i + \frac{1}{2} \hat{S}_{kj} \Gamma_{i \quad}^{kj} + \dots$$

cf. electrodynamics:  $p_i = \hat{p}_i - qA_i$

- Canonical field momentum  $\hat{\pi}^{ij\text{TT}}$  has **delta-corrections**:

$$\pi^{ij\text{TT}} = \hat{\pi}^{ij\text{TT}} + \frac{4\pi G}{m^2} \hat{p}_m \hat{p}_k \hat{S}^{lm} \delta_{kl}^{\text{TT}ij} \delta + \dots$$

can not be given in closed form explicitly

# PN Counting with Spin

for Hamiltonians

for maximally rotating objects:

$$S = \frac{Gm^2\chi}{c} \quad \chi = \mathbf{1}$$

order	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
$H^N$								
PM	$+ H^{1\text{PN}}$		$+ H^{2\text{PN}}$	$+ H^{2.5\text{PN}}$	$+ H^{3\text{PN}}$	$+ H^{3.5\text{PN}}$	$+ H^{4\text{PN}}$	$+ H^{4.5\text{PN}}$
SO		$+ H_{\text{SO}}^{\text{LO}}$		$+ H_{\text{SO}}^{\text{NLO}}$		$+ H_{\text{SO}}^{\text{N}^2\text{LO}}$	$+ H_{\text{SO}}^{\text{LO,R}}$	$+ H_{\text{SO}}^{\text{N}^3\text{LO}}$
$S_1^2$			$+ H_{S_1}^{\text{LO}}$		$+ H_{S_1}^{\text{NLO}}$		$+ H_{S_1}^{\text{N}^2\text{LO}}$	$+ H_{S_1}^{\text{LO,R}}$
$S_1 S_2$			$+ H_{S_1 S_2}^{\text{LO}}$		$+ H_{S_1 S_2}^{\text{NLO}}$		$+ H_{S_1 S_2}^{\text{N}^2\text{LO}}$	$+ H_{S_1 S_2}^{\text{LO,R}}$
spin <sup>3</sup>						$+ H_{S_3}^{\text{LO}}$		$+ H_{S_3}^{\text{NLO}}$
spin <sup>4</sup>							$+ H_{S_4}^{\text{LO}}$	
⋮								⋮

$H$  known    EOM known    for Black Holes    not known (yet)

Radiation field known to 2.5PN order, multipoles to 3PN order.

# Results for Spin Hamiltonians

shown for equal masses, circular orbits, and aligned spins

$$H_{\text{spin}} = H_{S_1 O} + H_{S_2 O} + H_{S_1^2} + H_{S_2^2} + H_{S_1 S_2} + H_{S^3} + H_{S^4} + \dots$$

LO

NLO

N<sup>2</sup>LO

$$H_{S_1 O} = S_1 L \left\{ \frac{7}{8r^3} + \frac{3}{r^4} \left[ -1 + \frac{5}{16} \frac{L^2}{r} \right] + \frac{1}{64r^5} \left[ 401 - \frac{751}{8} \frac{L^2}{r} - \frac{25}{16} \frac{L^4}{r^2} \right] + \dots \right\}$$

$$H_{S_1^2} = S_1^2 \left\{ -\frac{C_{ES^2}}{8r^3} + \frac{1}{16r^4} \left[ 6C_{ES^2} + 5 - \frac{17C_{ES^2} - 11}{4} \frac{L^2}{r} \right] + \dots \right\}$$

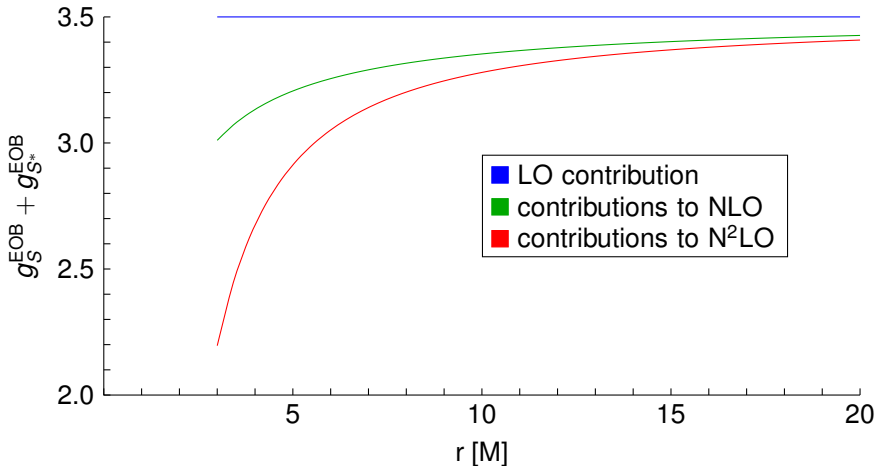
$$H_{S_1 S_2} = S_1 S_2 \left\{ -\frac{1}{4r^3} + \frac{1}{2r^4} \left[ 3 - \frac{7}{8} \frac{L^2}{r} \right] + \frac{1}{64r^5} \left[ -271 - 238 \frac{L^2}{r} + \frac{45}{8} \frac{L^4}{r^2} \right] + \dots \right\}$$

$$H_{S^3} = \frac{L}{16r^5} \left[ 5(S_1^3 + S_2^3) + \frac{105}{4} S_1 S_2 (S_1 + S_2) \right] + \dots \quad \text{yet only known}$$

$$H_{S^4} = -\frac{3}{32r^5} S_1 S_2 (S_1^2 + S_2^2) + \dots \quad \text{for black holes}$$

# Spin-Orbit: Gyro-Gravitomagnetic Ratios $g_S^{\text{EOB}} + g_{S^*}^{\text{EOB}}$

for equal masses and circular orbits, A. Nagar arXiv:1106.4349



# Conclusions & Outlook

## Conclusions:

- Spin (linear order): universal
- Quadrupole: internal structure, EOS
- Effects are small, but
  - accumulate during inspiral
  - become increasingly important in late inspiral
- Parameter space considerably increased!

## Possible Future Tasks:

- Spin part of radiation field at 3PN (and beyond)
- Spin Hamiltonians:
  - $H_{S3}^{LO}$  and  $H_{S4}^{LO}$  for (neutron) stars
  - $H_{S1}^{N^2LO}$  at 4PN
- More on tidal deformations
- Dynamical quadrupole  $L_Q \sim E_{\mu\nu} Q^{\mu\nu}$

Thank you for your attention

and for support by the German Research Foundation **DFG**  
and by the Fundação para a Ciência e a Tecnologia **FCT**