Constructing the action of Hořava–Lifshitz Gravity

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Beyond General Relativity

- *General Relativity is not a renormalizable theory and this means that beyond some scale it breaks down so that one cannot quantize it using conventional quantization techniques*
- *One could ask if once higher order curvature terms are included, renormalizability can be achieved*
- *Theories whose action includes invariants quadratic in the curvature are renormalizable (K. S. Stelle, Phys. Rev. D* **16**, *953 (1977)), but unfortunately this comes at a very high price, as such theories contain ghost degrees of freedom*

Hořava's Proposal

- *In 2009, Hořava proposed to modify the graviton propagator by adding higher order spatial derivatives without adding higher order time derivatives*
- *This prescription requires a splitting of spacetime into space and time and leads to Lorentz violations*
- *Lorentz invariance can be recovered at low-energy, or at least Lorentz violations in the infrared (IR) are requested to stay below current experimental constraints*

P. Hořava, Phys. Rev. D **79**, *084008 (2009)*

Foundations of the Theory

It is natural to consider such a theory within the framework of the ADM decomposition.
The full ADM metric is

$$ds^2 = -N^2 c^2 dt^2 + g_{ij}(dx_i + N_i dt)(dx_j + N_j dt) \tag{1}$$

The theory treats space and time on different footing then it is not invariant under the full set of diffeomorphisms but it is still invariant under the more restricted foliation-preserving diffeomorphisms

$$t \longrightarrow \tilde{t}(t), \quad x^i \longrightarrow \tilde{x}^i(t, x^i) \tag{2}$$

Foundations of the Theory

The most general action then is

$$S_H = S_K - S_V \tag{3}$$

where the kinetic term, which contains all of the time derivatives, is given by

$$S_K = \frac{2}{k^2} \int dt d^3x \sqrt{g} N \left(K_{ij} K^{ij} - \lambda K^2 \right) \tag{4}$$

K_{ij} *is the extrinsic curvature of the spacelike hypersurfaces,*

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i \right) \tag{5}$$

Foundations of the Theory

and the potential term is

$$S_V = \frac{k^2}{8} \int dt d^3x \sqrt{g} N\, V\left[g_{ij}, N\right] \tag{6}$$

- *Power counting renormalizability requires as a minimal prescription at least 6th order spatial derivatives in V*
- *The most general potential V with operators of dimensions up to 6, considering that the operators with odd dimensions are forbidden by spatial parity, contains tens of terms*
- *The theory also propagates a scalar degree of freedom (0-spin graviton)*

Foundations of the Theory

In the most general theory some of the terms that one can consider in the potential are

$$
\begin{aligned}
&(dim\ 2) \qquad && R,\ a_i a^i \\
&(dim\ 4) \qquad && R^2,\ R_{ij}R^{ij},\ R\nabla_i a^i,\ a_i \Delta a^i,\ (a_i a^i)^2,\ a_i a_j R^{ij},\ \dots \qquad (7) \\
&(dim\ 6) \qquad && (\nabla_i R_{jk})^2,\ (\nabla_i R)^2,\ \Delta R \nabla_i a^i,\ a_i \Delta^2 a^i,\ (a_i a^i)^3,\ \dots
\end{aligned}
$$

where $a_i = \partial_i lnN$.

D. Blas, O. Pujolas and S. Sibiryakov, Phys. Rev. Lett. **104**, *181302 (2010)*

Detailed Balance with Projectability

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2. *We can impose as an additional symmetry to the theory the so called "**Detailed Balance**": it requires that V should be derivable from a superpotential W as follows*

$$V = E^{ij} G_{ijkl} E^{kl} \tag{9}$$

where E^{ij} is given in term of a superpotential W as

$$E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W\,[g_{kl}]}{\delta g_{ij}} \tag{10}$$

The Superpotential W

The most general superpotential which contains all of the possible terms up to third order in spatial derivatives, which are invariant under foliation preserving diffeomorphisms is

$$W = \frac{1}{w^2}\int \omega_3(\Gamma) + \mu \int d^3x\sqrt{g}\,(R - 2\Lambda_W) \qquad (11)$$

where $\omega_3(\Gamma)$ *is the gravitational Chern-Simons term.*

The potential V *corresponding to this superpotential* W *is*

$$\begin{aligned} V \quad = \quad & \frac{4}{w^4}C_{ij}C^{ij} - \frac{4\mu}{w^2}\epsilon^{ijk}R_{il}\nabla_j R^l_k + \mu^2 R_{ij}R^{ij} \\ & - \frac{\mu^2}{(1-3\lambda)}\left(\frac{1-4\lambda}{4}R^2 + \Lambda_W R - 3\Lambda_W^2\right) \end{aligned} \qquad (12)$$

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3. *The scalar mode does not satisfy a sixth order dispersion relation and is not power-counting renormalizable (see later)*

T. P. Sotiriou, M. Visser and S. Weinfurtner, Phys. Rev. Lett. **102**, *251601 (2009)*

C. Appignani, R. Casadio and S. Shankaranarayanan, JCAP **1004**, *006 (2010)*

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Less obvious:

1. *The infrared behavior of the scalar mode is plagued by instabilities and strong coupling at unacceptably low energies*

C. Charmousis, G. Niz, A. Padilla and P. M. Saffin, JHEP **0908**, *070 (2009)*

D. Blas, O. Pujolas and S. Sibiryakov, JHEP **0910**, *029 (2009)*

Mass Scales in the Action

In order to get a clearer picture on the various scales involved in the action, we want to bring it to the form

$$S_H = \frac{M_{\rm pl}^2}{2}\int d^3x dt N\sqrt{g}\left\{K^{ij}K_{ij}-\lambda K^2+\xi R-2\Lambda+\frac{1}{M_4^2}L_4+\frac{1}{M_5^2}L_5+\frac{1}{M_6^4}L_6\right\} \tag{13}$$

Then we perform the following redefinitions of the couplings

$$M_{\rm pl}^2 = \frac{4}{k^2}, \qquad M_6^2 = \frac{w^2}{2}M_{\rm pl}^2, \qquad M_4^2 = \frac{M_{\rm pl}^4}{\mu^2}, \qquad \xi = \frac{\Lambda_W}{(1-3\lambda)M_4^2}, \tag{14}$$

where $M_{\rm pl}$, M_4 and M_6 have dimensions of a mass, whereas ξ is dimensionless.

Mass Scales in the Action

The action now takes the form

$$S_H = \frac{M_{\rm pl}^2}{2}\int dt d^3x\sqrt{g}N\Big[K_{ij}K^{ij} - \lambda K^2 + \xi R - 2\Lambda - \frac{1}{M_4^2}R_{ij}R^{ij} + \frac{1-4\lambda}{4(1-3\lambda)}\frac{1}{M_4^2}R^2 + \frac{2}{M_6^2 M_4}\epsilon^{ijk}R_{il}\nabla_j R^l_k - \frac{1}{M_6^4}C_{ij}C^{ij}\Big] \tag{15}$$

where the (bare) cosmological constant is

$$\Lambda = \frac{3}{2}\xi^2(1-3\lambda)M_4^2 \tag{16}$$

The Sign of the Cosmological Constant

If we want the theory to be close to General Relativity in the IR, then

$$\lambda, \xi \sim 1 \tag{17}$$

to high accuracy.

Therefore it is obvious that Λ has to be negative in this case!

The Size of the Cosmological Constant

The size of the cosmological constant is related to the size of the energy scale M_4 (suppressing the fourth order operators), at which Lorentz-violating effects will become manifest.
For Lorentz violations to have remained undetected in sub-mm precision tests, as an optimistic estimate one would need roughly

$$M_4 \geq 1 \div 10\mathrm{meV} \tag{18}$$

Considering this mildest constraint coming from purely gravitational experiments, the value of the cosmological constant would be (roughly)

$$\Lambda \sim 10^{-60} M_{\mathrm{pl}}^4 \tag{19}$$

There is at best a 60 orders of magnitude discrepancy between the value required by detailed balance and the observed value!

Projectable Version of the Theory without DB

T. P. Sotiriou, M. Visser and S. Weinfurtner, Phys. Rev. Lett. **102**, *251601 (2009)*

$$
\begin{aligned}
S_p &= \frac{M_{\rm pl}^2}{2}\int d^3x dt N\sqrt{g}\Big\{K^{ij}K_{ij} - \lambda K^2 - g_0\, M_{\rm pl}^2 - g_1 R - g_2\, M_{\rm pl}^{-2}\, R^2 \\
&- g_3\, M_{\rm pl}^{-2}\, R_{ij}R^{ij} - g_4\, M_{\rm pl}^{-4}\, R^3 - g_5\, M_{\rm pl}^{-4}\, R(R_{ij}R^{ij}) - g_6\, M_{\rm pl}^{-4}\, R^i{}_j R^j{}_k R^k{}_i \\
&- g_7\, M_{\rm pl}^{-4}\, R\nabla^2 R - g_8\, M_{\rm pl}^{-4}\, \nabla_i R_{jk}\, \nabla^i R^{jk}\Big\} \qquad (20)
\end{aligned}
$$

- *Parity violating terms have been suppressed*
- *Power-counting renormalizability is achieved*
- *The cosmological constant is controlled by* g_0 *and it is not restricted*
- *Strong coupling and instabilities plaguing the scalar mode at low energies*

K. Koyama and F. Arroja, JHEP **1003**, *061 (2010)*

T. P. Sotiriou, M. Visser and S. Weinfurtner, JHEP **0910**, *033 (2009)*

A Solution for these Problems

- *Abandoning Projectability: one can use not only the Riemann tensor of g_{ij} and its derivatives in order to construct invariants under foliation preserving diffeomorphisms, but also the vector a_i*

- *In the version without detailed balance this leads to a proliferation of terms ($\sim 10^2$), while here there is, remarkably, only one 2-dim operator one can add to the superpotential W in the version with detailed balance: $a_i a^i$*

New Superpotential and Action

The superpotential W then becomes

$$W = \frac{1}{w^2}\int \omega_3(\Gamma) + \int d^3x\sqrt{g}\left[\mu\left(R - 2\Lambda_W\right) + \beta\, a_i a^i\right] \tag{21}$$

where β is a new coupling. The total action now looks as

$$S_H = \frac{M_{\rm pl}^2}{2}\int dtd^3x\sqrt{g}N\Bigg\{K_{ij}K^{ij} - \lambda K^2 + \xi R - 2\Lambda + \eta\, a^i a_i$$
$$-\frac{1}{M_4^2}R_{ij}R^{ij} + \frac{1-4\lambda}{4(1-3\lambda)}\frac{1}{M_4^2}R^2 + \frac{2\eta}{\xi M_4^2}\left[\frac{1-4\lambda}{4(1-3\lambda)}Ra^i a_i - R_{ij}a^i a^j\right]$$
$$-\frac{\eta^2}{4\xi^2 M_4^2}\frac{3-8\lambda}{1-3\lambda}(a^i a_i)^2 + \frac{2}{M_6^2 M_4}\epsilon^{ijk}R_{il}\nabla_j R^l_k + \frac{2\eta}{\xi M_6^2 M_4}C^{ij}a_i a_j - \frac{1}{M_6^4}C_{ij}C^{ij}\Bigg\} \tag{22}$$

where $\Lambda = \frac{3}{2}\xi^2(1-3\lambda)M_4^2$ as before.

D. Vernieri and T. P. Sotiriou, Phys. Rev. D **85**, *064003 (2012)*

Linearization at Quadratic Order in Perturbations

Assuming that some resolution to the cosmological constant problem were to be found, we simply set $\Lambda = 0$ *in the action and we perturb it to quadratic order, considering only scalar perturbations*

$$N = 1 + \alpha, \qquad N_i = \partial_i y, \qquad g_{ij} = e^{2\zeta}\delta_{ij} \tag{23}$$

Finally the quadratic action reads

$$\begin{aligned} S_H^{(2)} &= \frac{M_{\rm pl}^2}{2}\int dtd^3x\left\{\frac{2(1-3\lambda)}{1-\lambda}\dot{\zeta}^2 + 2\xi\left(\frac{2\xi}{\eta}-1\right)\zeta\partial^2\zeta \right. \\ &- \left. \frac{2(1-\lambda)}{1-3\lambda}\frac{1}{M_4^2}(\partial^2\zeta)^2\right\} \end{aligned} \tag{24}$$

Dispersion Relation and Stability

The dispersion relation for the scalar is then given by

$$\omega^2 = \xi \left(\frac{2\xi}{\eta} - 1 \right) \frac{1-\lambda}{1-3\lambda} p^2 + \frac{1}{M_4^2} \left(\frac{1-\lambda}{1-3\lambda} \right)^2 p^4 \qquad (25)$$

For the scalar to have positive energy as well as for the spin-2 graviton one needs

$$\lambda < \frac{1}{3} \qquad \text{or} \qquad \lambda > 1 \qquad (26)$$

whereas classical stability requires that

$$c_\zeta^2 = \xi \left(\frac{2\xi}{\eta} - 1 \right) \frac{1-\lambda}{1-3\lambda} > 0 \qquad (27)$$

Dispersion Relation and Stability

- *For the spin-2 graviton to be stable* $\xi > 0$ *must be required, then one has*

$$2\xi > \eta > 0 \tag{28}$$

- *The coefficient of the* p^4 *term cannot lead to an instability at higher energies.*

- *The scalar satisfies a fourth, and not a sixth, order dispersion relation:*
  The renormalizability properties of the theory are compromised!

Recovering Power-Counting Renormalizability

- *Adding fourth order terms in W would lead to both sixth and eight order terms for the scalar, rendering the theory power-counting renormalizable.*
  The fourth order terms one could add in the superpotential W are

$$\begin{aligned} & R^2\,, \quad R^{\mu\nu}R_{\mu\nu}\,, \quad R\nabla^i a_i\,, \quad R^{ij}a_i a_j\,, \\ & R a_i a^i\,, \quad (a_i a^i)^2\,, \quad (\nabla^i a_i)^2\,, \quad a_i a_j \nabla^i a^j\,. \end{aligned} \tag{29}$$

- *After adding these terms and imposing parity invariance in total there would be 12 free couplings in the theory.*
  This is roughly an order of magnitude less than the number of couplings in the theory without detailed balance.

Conclusions

- *Hořava–Lifshitz gravity as an UV complete gravity theory*
- *Problems of the Projectable theory with or without Detailed Balance*
- *Solutions: Abandoning Projectability; Adding fourth order terms in W*
- *Results: Improved behaviour of the Scalar Graviton in the IR; Power-counting renormalizability of the Theory*
- *Shortcomings: Magnitude and Sign of the Bare Cosmological Constant*
- *Perhaps there could be a cancellation between Bare Cosmological Constant and Vacuum Energy, leaving behind a tiny residual that would account for the observed value*