Analytical solutions for black holes in cosmological background

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December 15, 2015

Motivation

- Analytical solution of Einstein equations.
- Black Holes in presence of self gravitating matter.
- Inhomogeneity in asymptotically cosmological spacetime

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- Black Holes in presence of self gravitating matter.
- Inhomogeneity in asymptotically cosmological spacetime
- Two competing effects:
  - Gravitationally bound object;
  - Expanding universe.
- Coupling between local effects and cosmological evolution
  - Astronomical observations;
  - Dynamical Horizons;
  - Accretion models;
  - Causal Structure.

Overview

The line element

- *Flat* McVittie solution in isotropic coordinates: [McVittie, MNRAS **93**,325 (1933)]

$$ds^2 = -\frac{\left(1-\frac{m}{2ax}\right)^2}{\left(1+\frac{m}{2ax}\right)^2}\mathrm{d}t^2 + a^2\left(1+\frac{m}{2ax}\right)^4\left(\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2\right)\,.$$

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UFABC

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- If $m = 0$: FLRW metric.
- If $a(t)$ is constant: Schwarzschild metric.

Sources

- Comoving perfect fluid with homogeneous density: $\rho(t) = \frac{3}{8\pi}H^2$
- Inhomogeneous pressure:

$$p(t,x) = \frac{1}{8\pi}\left[H^2\frac{-5m+2ax}{m-2ax} + 2\frac{\ddot{a}}{a}\frac{m+2ax}{m-2ax}\right] .$$

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- Lagrangian source: Einstein gravity coupled with Cuscuton field [Abdalla, Afshordi, Fontanini, Guariento, Papantonopoulos, PRD **89**, 104018 (2014)]:

$$S_\phi = \int \mathrm{d}^4x\sqrt{-g}\left[\mu^2\sqrt{-g^{\alpha\beta}\phi_{;\alpha}\phi_{;\beta}} - V(\phi)\right].$$

- Imposing the null energy condition on the sources implies $\dot{H}(t) \leq 0$.

Areal radius coordinate

- It is convenient to use the areal radius: $r = a\left(1 + \frac{m}{2ax}\right)^2 x$
- The solution has two branches:

$$2ax = \left(r - m \pm \sqrt{(r-m)^2 - m^2}\right) .$$

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- McVittie line element – areal radius as a coordinate [Kaloper, Kleban, Martin, PRD **81**,104044 (2010)]:

$$\mathrm{d}s^2 = -R^2 + \left[\frac{\mathrm{d}r}{R} - Hr\mathrm{d}t\right]^2 + r^2\mathrm{d}\Omega^2 ,$$

with $R = \sqrt{1 - \frac{2m}{r}}$.

The singularity

- Singularity $r_* = 2m \Rightarrow R(r_*) = 0$ and $x = \frac{m}{2a}$:
  - Ricci scalar diverges unless $\dot{H} = 0$:

$$\mathcal{R} = 12H^2 + 6\dot{H}\left(\frac{m+2ax}{m-2ax}\right) .$$

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- r_* is at the past of all events since

$$\frac{\mathrm{d}(r - r*)}{\mathrm{d}t} = RrH + \mathcal{O}(R^2) > 0 ,$$

for either ingoing or outgoing null rays.

Apparent horizons

- Apparent horizons – zeros of expansion of null radial geodesics:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{\pm} = R(R \pm Hr) = 0\,.$$

- Only ingoing geodesics $(-)$ can have solutions

$$1 - \frac{2m}{r} - H^2 r^2 = 0\,.$$

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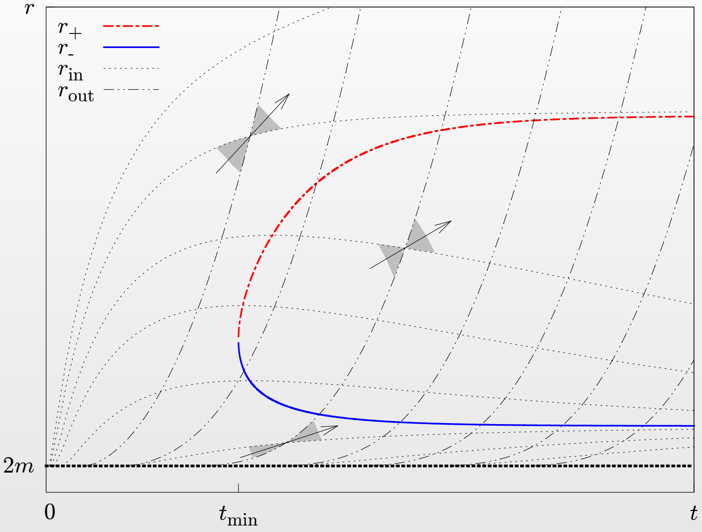
- For $\frac{1}{3\sqrt{3}} > mH > 0$, there are two positive solutions:
    - r_+ – "outer" (cosmological) horizon;
    - r_- – "inner" horizon.
- The apparent horizons r_+ *and* r_- are *anti-trapping*.

Light cones



Geodesic completeness

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$H(t) \to H_0 > 0$

- r_+ and r_- tend to Schwarzschild-de Sitter horizons values;
- r_- is an anti-trapping horizon at finite times, but for $t \to \infty$, $r_- \to r_\infty$ and the surface $(t \to \infty, r = r_\infty)$ is a *trapping horizon*;
- The surface $(t \to \infty, r = r_\infty)$ lies at finite distance along null geodesics and it is traversable;
- This McVittie metric describes a universe that contains a black hole.

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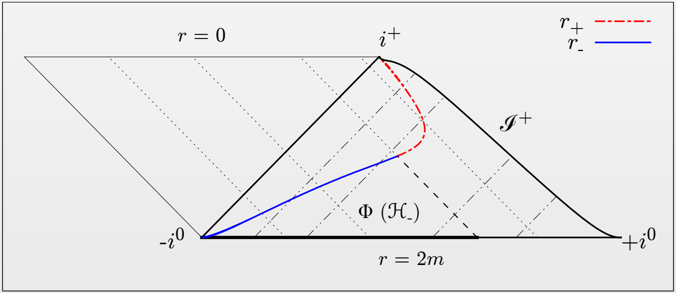
$H(t) \to 0$

- r_- tends to its Schwarzschild value $r_\infty = 2m$;
- The surface $(t \to \infty, r = r_\infty)$ is at a finite distance along null geodesics;
- There is disagreement in the literature if that surface is a traversable event horizon or a singularity;
  [Kaloper, Kleban, Martin, PRD **81**,104044 (2010)][Lake, Abdelqader, PRD **84**,044045 (2011)]

Penrose diagrams

- Penrose diagram plus analytical continuation with Schwarzschild-de Sitter trapped region for $H_0 > 0$.
- It describes a black hole spacetime.



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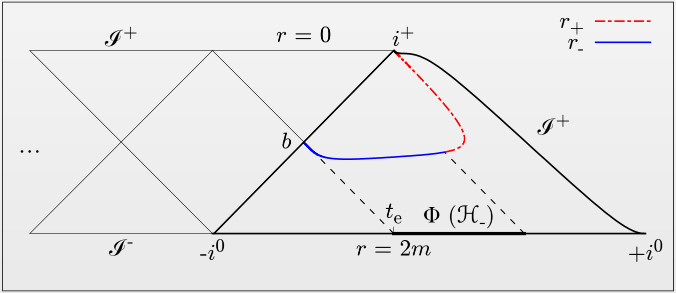

The causal structure theorem

- We can distinguish between the two cases by studying the flow of ingoing null radial geodesics
- We approximate the null geodesics by known curves from above and below in order to tell if they fail to cross the r_- horizon at large times.

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- Let be $B = \frac{1-3H_0^2 r_\infty^2}{2r_\infty} > 0$, $\Delta H = H(t) - H_0$ and

$$F_+ = \int_{t_0}^{t} e^{(B-\sigma)u} \Delta H(u) \mathrm{d}u \,,$$

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[AMS, Fontanini, Guariento, PRD 87,064030 (2013)] [AMS, Guariento, C. Molina, PRD **91**, 084043 (2015)]

The gMcVittie metric

UFABC

- In isotropic coordinates [Faraoni, Jacques, PRD **76**, 063510 (2007)]

$$\mathrm{d}s^2 = -\frac{\left(1-\frac{m(t)}{2a(t)x}\right)^2}{\left(1+\frac{m(t)}{2a(t)x}\right)^2}\mathrm{d}t^2 + a(t)^2\left(1+\frac{m(t)}{2a(t)x}\right)^4\left(\mathrm{d}x^2 + x^2\mathrm{d}\Omega^2\right) \,.$$

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- Using the areal radius coordinate

$$ds^2 = -R^2 + \left[\frac{dr}{R} - \mathcal{H}(t,r)\, r dt\right]^2 + r^2 d\Omega^2 ,$$

with

$$\mathcal{H}(t,r) = H + \frac{\dot{m}}{m}\left(\frac{1}{R} - 1\right) .$$

- We assume here $M(t) \equiv \frac{\dot{m}}{m} > 0$ — accretion case.

Sources

- Comoving fluid with a radial heat flow

$$T^{\mu\nu} = T^{\mu\nu}_{\text{pf}} - \zeta h^{\mu\nu}\nabla_\gamma u^\gamma - \chi(h^{\mu\gamma}u^\nu + h^{\nu\gamma}u^\mu)q_\gamma \,,$$
$$h^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu \,; \quad q_\gamma = \nabla_\gamma \mathcal{T} + \mathcal{T} u^\alpha \nabla_\alpha u_\gamma \,.$$

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- The energy density and temperature profile given by

$$\rho(t,x) = \frac{3}{8\pi}\left[\frac{2\dot{m}}{2ax - m} + H\right]^2 \, ,$$
$$\mathcal{T}(t,x) = \frac{1}{\sqrt{-g_{tt}}}\left[\mathcal{T}_\infty(t) + \frac{\dot{m}}{4\pi\chi m}\ln(\sqrt{-g_{tt}})\right] \, .$$

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- Lagrangian source: beyond Horndeski or G^3 theory [Afshordi, Fontanini, Guariento, PRD **90**, 084012 (2014)]

The singularity

UFABC

- Located at $r_* = 2m(t)$.
- It is spacelike if $H(t) - M(t) \neq 0\,, \forall t > 0$. Null if $H = M$ (Thakurta solution).

$$n_\mu = -2\dot{m}\mathrm{d}t + \mathrm{d}r\,, n^\mu n_\mu = -4m(H - M)^2\,.$$

- It is in the past of all events if $H(t) - M(t) > 0$:

$$\frac{\mathrm{d}(r - r_*)}{\mathrm{d}t} = Rr(H - M) + \mathcal{O}(R^2) > 0\,.$$

- Condition $H > M$ guarantees a big-bang-like singularity.

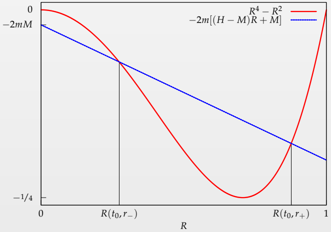
Apparent horizons

UFABC

- Ingoing null geodesics governed by fourth order polynomial in R:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)_{\text{in}} = \frac{1}{1-R^2}\left\{-R^4 + R^2 - 2m\left[(H-M)\,R + M\right]\right\} = 0\,,$$

- There exist at most two positive real solutions provided $M > 0$, $H - M > 0$:



- r_+ and r_- are anti-trapping horizons for finite t.
- If $m_0 \equiv \lim_{t\to\infty} m(t)$ is finite, then r_+ and r_- converges to Schwarzschild-de Sitter horizons values.

Geodesic completeness

- Under the following assumptions:
  - $m(t) > 0\,, \quad \forall t > 0$ — positive mass function;
  - $m_0 \equiv \lim_{t\to\infty} m(t) > 0\,,$ — bounded mass function;
  - $H_0 \equiv \lim_{t\to\infty} H(t) > 0\,,$ — de-Sitter accelerated expansion;
  - $\frac{1}{3\sqrt{3}} > m_0 H_0 > 0\,,$ — two non-degenerate apparent horizons;
  - $\dot{M}(t) > 0\,, \forall\, t > 0\,,$ — mass grows monotonically
  - $H(t) - M(t) > 0\,, \forall\, t > 0\,,$ — presents a Big-Bang-like singularity.

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  - $H(t) - M(t) > 0\,, \forall\, t > 0\,,$ — presents a Big-Bang-like singularity.
- All the gMcVittie spacetimes are geodesically incomplete towards the surface $(t \to \infty, r_-(t) \to r_\infty)$.
- This surface is traversable and can be continued by a patch of a Schwarzschild-de Sitter metric with parameters m_0 and H_0.

  [AMS, Guariento, C. Molina, PRD **91**, 084043 (2015)]

Penrose diagrams

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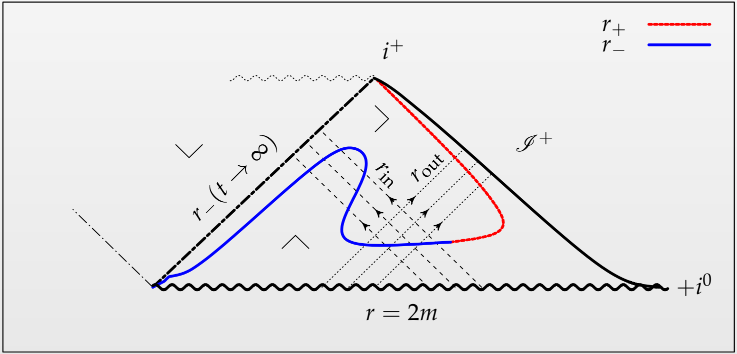
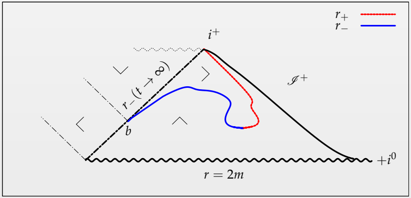
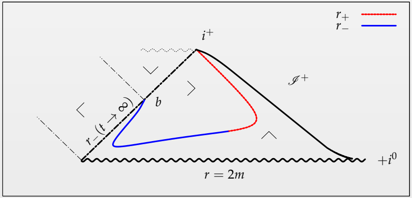
- Single cosmological black hole;
- Analytical continuation – trapped region of Schwarzschild-de Sitter spacetime.



Penrose diagrams

- Black hole/white hole pair;
- Analytical continuation – trapped region plus regular region of Schwarzschild-de Sitter spacetime.





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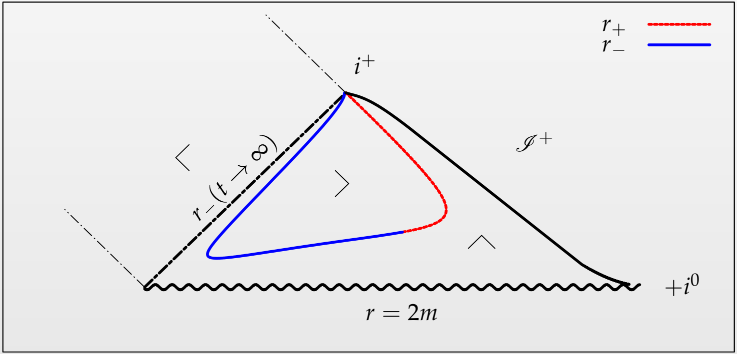

Causal structure theorem – gMcVittie

- Consider the function $\xi(t) > 0\,,\ \forall t \in (0,\infty)\,,$ given by

$$\xi(t) = r_\infty \Delta H(t) + \frac{m(t) - m_0}{r_\infty^2 H_0} + \frac{M}{H_0}(1 - r_\infty H_0) > 0\,,$$

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- We distinguish two possibilities on $\dot{\xi}$:
  a. $\dot{\xi}(t) < 0$ for large t — includes standard McVittie as $\dot{\xi} = \dot{H}$;
  b. $\dot{\xi}(t) > 0$ for large t — exclusively in gMcVittie, when $\dot{M}$ dominates.

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- We define

$$F_+ = \int_{t_0}^{t} e^{(B-\sigma)u} \xi(u) \mathrm{d}u\,,$$
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(1)

Causal structure theorem – gMcVittie

UFABC

$\dot{\xi} < 0$

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[AMS, Guariento, C. Molina, PRD **91**, 084043 (2015)]

Table : Possible asymptotic structures of gMcVittie spacetimes.

	$\dot{\xi}(t) \to 0^-$	$\dot{\xi}(t) \to 0^+$
$\left\| \int^{\infty} e^{(B-\sigma)u} \xi(u) \mathrm{d}u \right\| < \infty$	black hole and white hole	black hole and white hole
$\left\| \int^{\infty} e^{(B+\sigma)u} \xi(u) \mathrm{d}u \right\| \to \infty$	black hole only	white hole only

Conclusions

On the McVittie metric

- The flat McVittie solution with $H_0 > 0$ does describe cosmological black holes.
- It is a solution of gravity coupled with cuscuton field.
- The causal structure depends on the asymptotic behavior of $H(t)$ and not only on its limit value at time infinity.

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On the gMcVittie metric

- Under physically reasonable conditions, the gMcVittie metric describes cosmological black holes with time-dependent mass.
- It is a solution of a Horndeski Lagrangian.
- The causal structure depends on the asymptotic behavior of the $m(t)$ and $H(t)$ functions and a new case of a single white hole is possible.