Fundação para a Ciência e a Tecnologia
MINISTÉRIO DA CIÊNCIA, TECNOLOGIA E ENSINO SUPERIOR

Thermodynamics of a rotating thin shell in the BTZ spacetime

Masato Minamitsuji
(CENTRA, IST, U-Lisboa)

J.P.S. Lemos, F. J. Lopes and M. Minamitsuji, IJMPD 24, 9, 1542022 (2015)
J.P.S. Lemos, F. J. Lopes, M. Minamitsuji and J.V. Rocha, PRD 92, 084012 (2015)
J.P.S. Lemos, M. Minamitsuji and O.B. Zaslavskii, in progress.

A Thin Shell Approach to BH Thermodynamics

- The origin of the Bekenstein-Hawking entropy is still one of the greatest mysteries in the modern theoretical physics.

Bekenstein (73), Bardeen, Carter & Hawking (73) , Hawking (75)

$$S = \frac{A_+}{4G}$$

A_+: Area of the event horizon

$c = \hbar = k_B = 1$

- Since BHs are vacuum solutions, while our naïve concepts of "entropy" are based on quantum properties of matter, it would be useful to study whether black hole thermodynamics could emerge when one compresses matter within its own gravitational radius.

- One of the simplest systems of collapsing matter is the infinitesimally thin shells where self-gravitating matter is confined, placed in an otherwise vacuum spacetime.

- The previous studies on thermodynamics of a thin matter shell have recovered the Bekenstein-Hawking entropy, when the shell is taken to its gravitational radius and the temperature of the shell coincides with Hawking temperature.

  - Outer Schwarzschild and inner Minkowski in the (3+1) dimensions.

    Martinez (96)

  - Outer Reissner-Nortstroem and inner Minkowski in the (3+1) dimensions.

    Lemos, Quinta and Zaslavskii (15)

  - Outer (non-rorating) BTZ and inner AdS in the (2+1)-dimensions.

    Lemos and Quinta (14)

- We study the thermodynamic properties of a thin matter shell in the outer rotating BTZ and inner AdS spacetimes in the (2+1) dimensions, as the simple analogue of the (3+1)-dimensional Kerr spacetime.

BTZ Spacetime in the (2+1) dimensions

Banados, Teitelboim & Zanelli (92)

- The exact stationary solution of GR in the (2+1) dimensions with negative cosmological constant $\Lambda < 0$.

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} dt^2 + \frac{\ell^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi - \frac{r_+ r_-}{\ell r^2} dt \right)^2$$

$$\ell = \sqrt{-\frac{1}{\Lambda}}$$

ADM mass $r_+^2 + r_-^2 = 8G\ell^2 m$

Angular momentum $r_+ r_- = 4G\ell\mathcal{J}$

$$r_+ \geq r_- \Leftrightarrow m \geq \frac{\mathcal{J}}{\ell}$$

- Similarities with the Kerr solution in (3+1) dimensions

  - Uniqueness - Ergo region $r < (r_+^2 + r_-^2)^{\frac{1}{2}}$

interesting as the simple analogue of the Kerr spacetime.

A Timelike Thin Shell in the (2+1) Dimensions

- We introduce a timelike thin matter shell located at $r = R$.
- The outer region ($r > R$) is taken to be the rotating BTZ.

  $R > r_+$ $\Rightarrow$ the shell is always timelike and located outside the horizon.
- The inner region ($r < R$) is taken to be the (2+1)-dim AdS, so that no singularity and horizon are formed inside the shell.

$$ds^2_{(I)} = -f_{(I)}(r)^2 dt_I^2 + g_{(I)}(r)^2 dr^2 + r^2 \left(d\phi + h_{(I)}(r) dt_{(I)}\right)^2$$

$$I = o/i$$

outer inner

$$f_{(o)}(r) = \frac{\sqrt{(r^2 - r_+^2)(r^2 - r_-^2)}}{\ell r}, \quad g_{(o)}(r) = \frac{1}{f_{(o)}(r)}, \quad h_o(r) = -\frac{r_+ r_-}{\ell r^2}$$

$$f_{(i)}(r) = \frac{r}{\ell}, \quad g_{(i)}(r) = \frac{1}{f_{(i)}(r)}, \quad h_{(i)}(r) = 0.$$

- In order to match two regions, the shell at $r = R$ (and the inner region) must corotate with the outer BTZ region   Poisson (07)

$$d\psi = d\phi + h_{(I)}(R)dt_{(I)}$$

$$\Rightarrow \quad ds^2_{(I)} = -f_{(I)}(r)^2 dt^2_{(I)} + g_{(I)}(r)^2 dr^2 + r^2\left(d\psi + \bar{h}_{(I)}(r)dt_{(I)}\right)^2$$

$$\bar{h}_{(I)}(r) = h_{(I)}(r) - h_{(I)}(R) \qquad \bar{h}_{(I)}(R) = 0$$

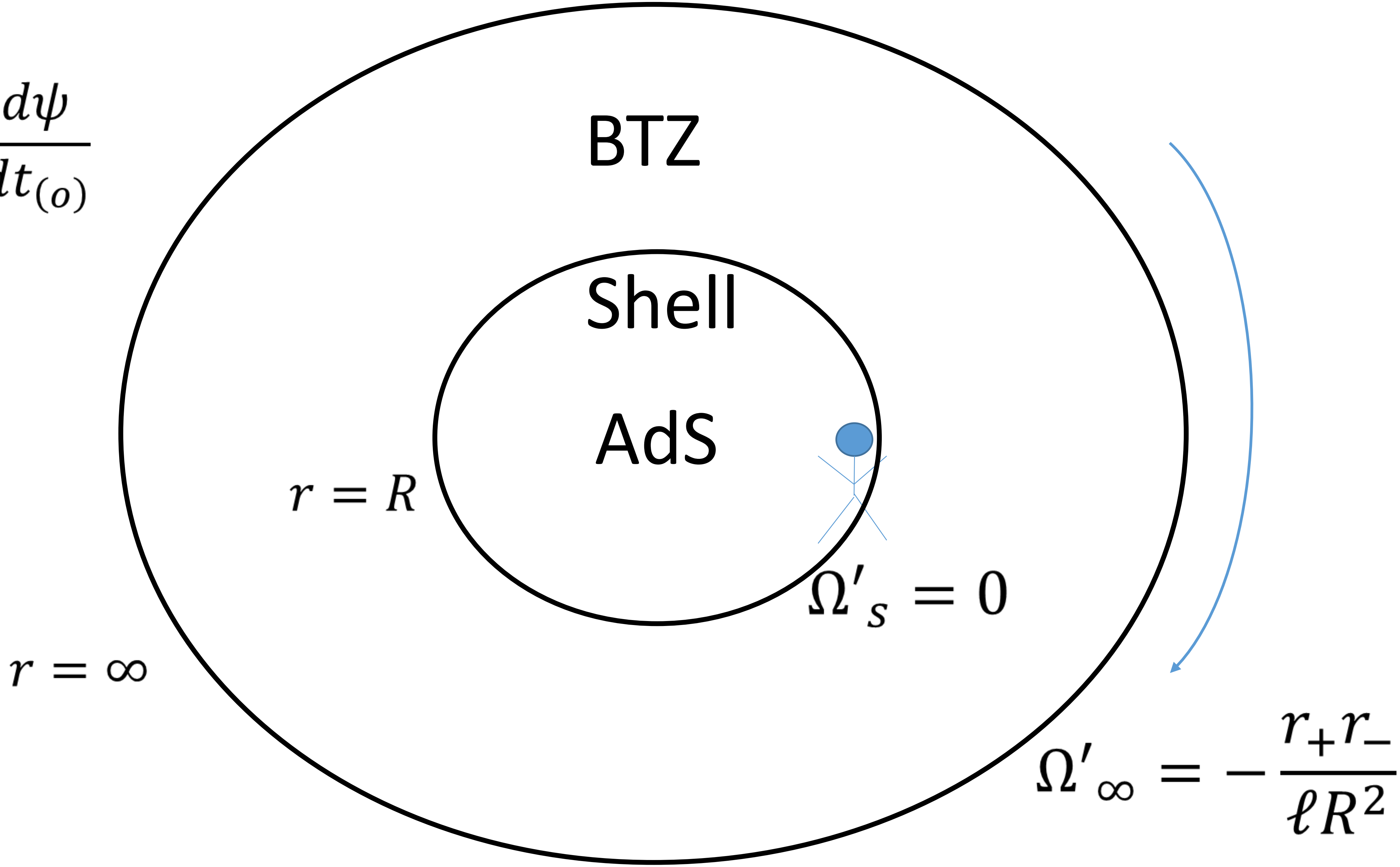
- At the position of the shell $r = R$, $\bar{h}_{(I)}(R) = 0$, and the induced line element on the shell is given by

$$ds^2_{\Sigma} = h_{ab}dy^a dy^b = -d\tau^2 + R^2 d\psi^2, \quad R = R(\tau)$$

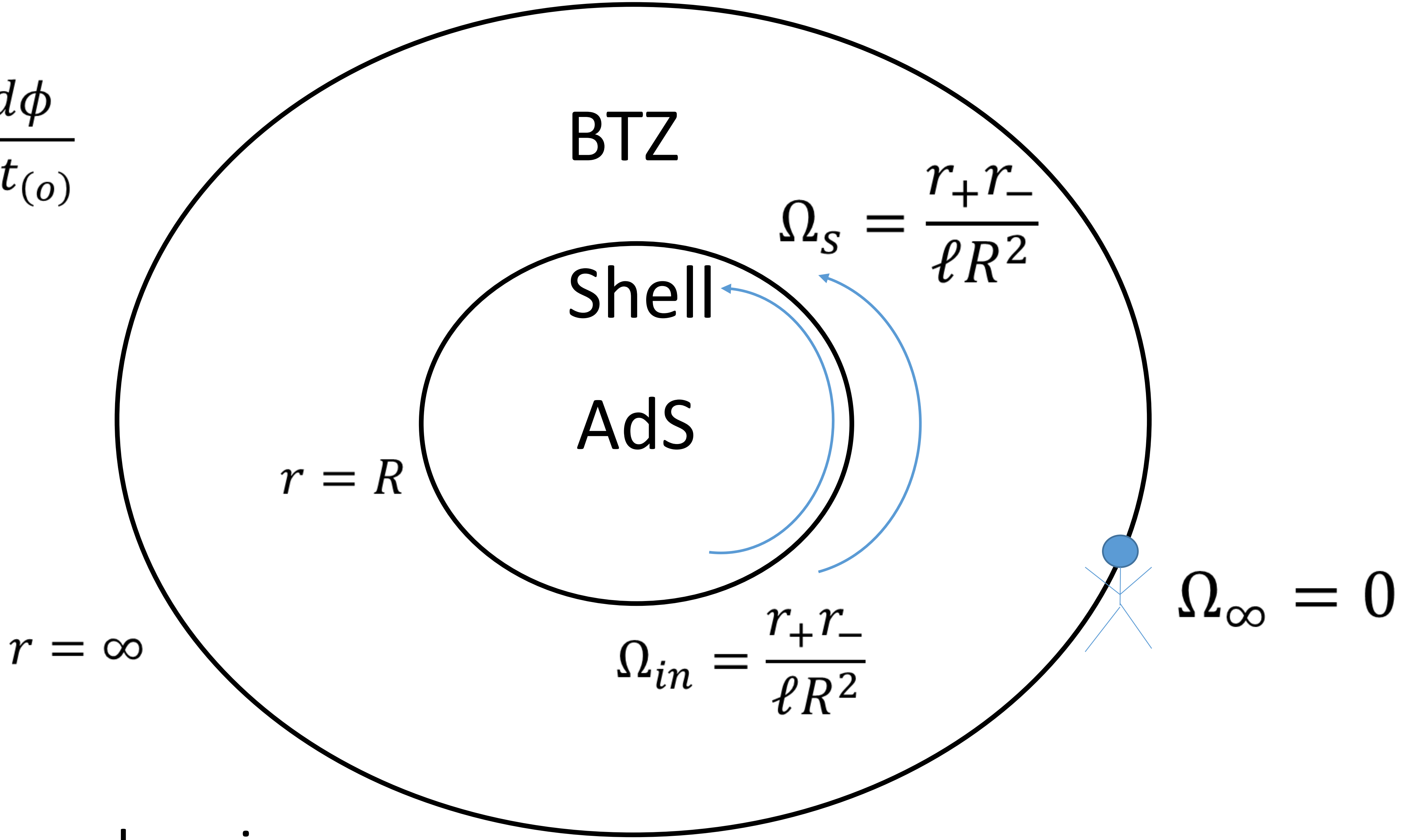
$$d\tau = \sqrt{f_{(o)}(R)^2 dt^2_{(o)} - g_{(o)}(R)^2 dR^2} = \sqrt{f_{(i)}(R)^2 dt^2_{(i)} - g_{(i)}(R)^2 dR^2}$$

- We are interested in the quasistatic process and assume that $\dot{R} = \ddot{R} = 0$.

$$\Omega' = \frac{d\psi}{dt_{(o)}}$$



$$\Omega = \frac{d\phi}{dt_{(o)}}$$



frame dragging

Imperfect Fluid on the Shell

Lemos, Lopes and Minamitsuji (15)

- The rotating thin shell is supported by an imperfect fluid

Energy density

$$\sigma = \frac{1}{8\pi G\ell}\left(1 - \frac{1}{R^2}\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}\right)$$

Pressure

$$p = \frac{1}{8\pi G\ell}\left(\frac{R^4 - r_+^2 r_-^2}{R^2\sqrt{(R^2 - r_+^2)(R^2 - r_-^2)}} - 1\right)$$

Angular-momentum flux density

$$j = \frac{r_+ r_-}{8\pi G\ell R}$$

$p \geq \sigma$ $\Rightarrow$ The DEC is violated except for the extremal case.

- The local proper mass and angular momentum of the shell

$$M = 2\pi R\sigma \qquad J = 2\pi Rj = \frac{r_+ r_-}{4G\ell}$$

- ADM mass and angular momentum measured in the outer region

$$m = \frac{RM}{\ell} - 2GM^2 + \frac{2G}{R^2}J^2 \qquad \mathcal{J} = J$$

Thermodynamcs of a Rotating Matter Shell

Lemos, Lopes, Minamitsuji and Rocha (15)

- We assume that the shell is in thermal equilibrium with a locally measured temperature $T(= 1/\beta)$ and entropy S.
- The entropy S can then be expressed as a function of the independent state variables, the proper mass M, the area of the shell $A(= 2\pi R)$ and the angular momentum J.
- The 1st law of thermodynamics

$$TdS = dM + p\,dA - \Omega\,dJ$$

$$\Rightarrow \quad dS = \beta\,dM + 2\pi p\beta\,dR - \beta\Omega\,dJ.$$

- The integration of the 1st law to yield $S(M, R, J)$ can be performed once the equations of state are specified.

$$p = p(M, R, J), \quad \beta = \beta(M, R, J), \quad \Omega = \Omega(M, R, J)$$

- For M, J, p, we use the expressions obtained from the junction conditions.

- β and Ω play the role of integration factors, which must be specified in order to obtain an exact expression of the entropy. However , the choice of these function can be specified is constrained by the integrability conditions that directly follow from the 1st law.

$$\left(\frac{\partial \beta}{\partial R}\right)_{J,M} = 2\pi \left(\frac{\partial (\beta p)}{\partial M}\right)_{J,R} \qquad \left(\frac{\partial \beta}{\partial R}\right)_{J,M} = 2\pi \left(\frac{\partial (\beta p)}{\partial M}\right)_{J,R}$$

$$2\pi \left(\frac{\partial (\beta p)}{\partial J}\right)_{R,M} = -\left(\frac{\partial (\beta \Omega)}{\partial R}\right)_{J,M}$$

- The redshift function $k(r_+, r_-, R) = \frac{R}{\ell}\sqrt{\left(1-\frac{r_+^2}{R^2}\right)\left(1-\frac{r_-^2}{R^2}\right)}$
- The pressure equation of state

$$p(M,R,J) = \frac{1}{8\pi G\ell}\left(\frac{R^4 - r_+^2(M,R,J)\,r_-^2(M,R,J)}{R^3\ell k(r_+(M,R,J), r_-(M,R,J), R)} - 1\right) = \frac{MR^3 - 4G\ell J^2}{2\pi R^3(R-4G\ell M)}$$

- The temperature equation of state

$$\frac{1}{\beta}\left(\frac{\partial\beta}{\partial R}\right)_{r_+,r_-} = \frac{1}{\beta}\left\{\left(\frac{\partial\beta}{\partial R}\right)_{M,J} + \left(\frac{\partial M}{\partial R}\right)_{r_+,r_-}\left(\frac{\partial\beta}{\partial M}\right)_{R,J}\right\} = \frac{1}{\beta}\left\{2\pi\beta\left(\frac{\partial p}{\partial M}\right)_{R,J} + \left[2\pi p + \left(\frac{\partial M}{\partial R}\right)_{r_+,r_-}\right]\left(\frac{\partial\beta}{\partial M}\right)_{R,J}\right\}$$

$$= 2\pi\left(\frac{\partial p}{\partial M}\right)_{J,R} = \frac{R^4 - r_+^2 r_-^2}{R(R^2-r_+^2)(R^2-r_-^2)} = \frac{1}{k}\left(\frac{\partial k}{\partial R}\right)_{r_+,r_-}$$

$\Longrightarrow$ Tolman relation for the thin shell in the BTZ spacetime

$$\beta(r_+, r_-, R) = k(r_+, r_-, R)\,b(r_+, r_-)$$

integration constant

$b(r_+, r_-)$ is the (inverse) temperature at $R = \left\{\frac{1}{2}\left(\ell^2 + r_+^2 + r_-^2 + \sqrt{\ell^4 + (r_+^2 - r_-^2)^2 + 2\ell^2(r_+^2 + r_-^2)}\right)\right\}^{\frac{1}{2}}$.

- The angular velocity equation of state

$$\left(\frac{\partial p}{\partial J}\right)_{M,R}+\Omega\left(\frac{\partial p}{\partial M}\right)_{J,R}=p\left(\frac{\partial \Omega}{\partial M}\right)_{J,R}-\frac{1}{2\pi}\left(\frac{\partial \Omega}{\partial R}\right)_{M,J}$$

$$=-\frac{1}{2\pi}\left\{\left(\frac{\partial \Omega}{\partial R}\right)_{M,J}+\left(\frac{\partial M}{\partial R}\right)_{r_+,r_-}\left(\frac{\partial \Omega}{\partial M}\right)_{J,R}\right\}=-\frac{1}{2\pi}\left(\frac{\partial \Omega}{\partial R}\right)_{r_+,r_-}$$

$$\Rightarrow \quad \left(\frac{\partial(\Omega\beta)}{\partial R}\right)_{r_+,r_-}=-2\pi\beta\left(\frac{\partial p}{\partial J}\right)_{M,R}=\frac{2r_+r_-b(r_+,r_-)}{\ell R^3}$$

$$\Rightarrow \quad \Omega(r_+,r_-,R)=\frac{r_+r_-}{\ell k(r_+,r_-,R)}\left(c(r_+,r_-)-\frac{1}{R^2}\right)$$

Integration constant

- In summary, we have found the three equations of state, $p(r_+,r_-,R)$, $\beta(r_+,r_-,R)$ and $\Omega(r_+,r_-,R)$ which are necessary to determine the entropy of the shell.

The Entropy of the Shell

- By changing the variable from (M, R, J) to (R, r_+, r_-) with

$$M(r_+, r_-, R) = \frac{1}{4G}\left(\frac{R}{\ell} - k(r_+, r_-, R)\right) \qquad J(r_+, r_-) = \frac{r_+ r_-}{4G\ell}$$

and substituting the equations of state $p(r_+, r_-, R), \beta(r_+, r_-, R)$ and $\Omega(r_+, r_-, R)$

$$dS = \frac{b}{8G\ell^2}\left[\left(1 - r_-^2 c(r_+, r_-)\right) dr_+^2 + \left(1 - r_+^2 c(r_+, r_-)\right) dr_-^2\right].$$

- The two integration constants must satisfy the integrability condition

$$\frac{\partial b}{\partial r_+^2} - \frac{\partial b}{\partial r_-^2} = \frac{\partial (bc)}{\partial \log(r_+^2)} - \frac{\partial (bc)}{\partial \log(r_-^2)}$$

Any choice of $b = f(r_+^2 + r_-^2)$ and $bc = g(r_+^2 r_-^2)$ will satisfy the integrability conditions.

- In generic cases, in order to obtain the specific expression for the entropy, we need to choose $b(r_+, r_-)$ and $c(r_+, r_-)$.

- The entropy S is a function of r_+ and r_- only

$$S = S(r_+, r_-)$$

and hence a function of (M, R, J) through $r_\pm(M, R, J)$,

$$S(M, R, J) = S\big(r_+(M, R, J), r_-(M, R, J)\big)$$

- Shells with the same r_+ and r_-, namely the same ADM mass m and angular momentum J but at a different position R, have the same entropy.

$\Rightarrow$ Thus an observer measuring m and J cannot distinguish shells with different radius by measuring the entropy.

The Thin Shell and the Black Hole Limit

- As the inverse temperature equation of state $b(r_+, r_-)$, we choose

$$b(r_+, r_-) = b_+ \gamma$$

$$b_+ = 2\pi\ell^2 \frac{r_+}{r_+^2 - r_-^2}$$ (Inverse) Hawking temperature of a BTZ BH

$\gamma > 0$ Constant which depends on the properties of matter

- There is a family of solutions for the angular velocity equation of state, $c(r_+, r_-)$ which satisfies the integrability condition. Here, we choose the particular solution

$$c(r_+, r_-) = \frac{1}{r_+^2}$$

which makes the thermodynamic angular velocity, Ω, vanish when the shell is pushed to the gravitational radius, $R \to r_+$.

- By substituting these functions into the 1st law, we obtain the differential for the entropy of the shell

$$dS = \frac{\gamma}{4G} dA_+ \qquad A_+ = 2\pi r_+$$

- By integrating this, the entropy of the shell is given by $S = S_0 + \frac{\gamma}{4G} A_+$, where S_0 is an integration constant. Requiring that when the shell is absent - or equivalently $M = 0$ and $J = 0 (\Leftrightarrow r_+ = r_- = 0)$, we fix $S_0 = 0$,

$$S = \frac{\gamma}{4G} A_+$$

which shows that entropy of the shell depends on (M, R, J) only through r_+.

- As the shell approaches its gravitational radius, $R \to r_+$, quantum effects would be inevitably present and their backreaction would invalidate our classical treatment. Therefore, we must choose $\gamma = 1$ or $b = b_+$

$$\Rightarrow \; S = \frac{A_+}{4G}$$

When we push the shell to its gravitational radius,
the entropy coincides with the Bekenstein-Hawking entropy.

- Thus in the case of the nonextremal shells, the Bekenstein-Hawking entropy is recovered if the temperature of the collapsing shell coincides with the Hawking temperature.

The Extremal Rotating Thin Shells

Lemos, Minamitsuji and Zaslavskii, in progress

- The extremal rotating thin shells $r_- = r_+$ have very distinct properties.
- The local mass and angular momentum of extremal rotating shells

$$M = 2\pi R\sigma = \frac{r_+^2}{4G\ell R} \qquad J = MR \qquad p = \frac{M}{2\pi R}$$

- The 1st of thermodynamics $v = \Omega R$: the velocity equation of state

$$\Rightarrow dS = \frac{\beta}{R}(1-v)d(MR) \Rightarrow dS = \frac{r_+}{2G\ell}\frac{\beta}{R}(1-v)dr_+$$

- The integrability condition yields $\frac{r_+}{2G\ell}\frac{\beta}{R}(1-v) = s(r_+)$

- The entropy S is only the function of the gravitational radius

$$dS = s(r_+)dr_+ \Rightarrow S = S(r_+), \quad R > r_+$$

- Since for the extremal shell the integrability condition has nothing to do with the Tolman relation, the temperature equation of state may be expressed as

$$\beta(r_+, R) = b(r_+, R) k(r_+, R) \qquad k(r_+, R) = \frac{R}{\ell}\left(1 - \frac{r_+^2}{R^2}\right)$$

- The velocity equation of state is then given by the integrability condition

$$v(r_+, R) = \frac{R}{\ell k(r_+, R)}\left(c(r_+, R) - \frac{r_+^2}{R^2}\right) \qquad c(r_+, R) = 1 - \frac{2G\ell^2}{r_+ b(r_+, R)} s(r_+)$$

- For the explicit computation of the entropy b and c must be specified as

$$\frac{r_+}{2G\ell^2} b(r_+, R)\big(1 - c(r_+, R)\big) = s(r_+)$$

- The limit of the extremal shell to the extremal black hole, $R \to r_+$, is taken by choosing $b = \infty$ $(T = 0)$ and $c = 1$ $(v = 1)$, keeping $b(1 - c)$ finite.
- Thus the entropy of the extremal shell can be any well-behaved function of r_+ .

$$S = S(r_+), \qquad R = r_+$$

Summary

- For a nonextremal shell Bekenstein-Hawking entropy is recovered, when the shell is pushed to its gravitational radius and its temperature is taken to the Hawking temperature.
- For an extremal shell, the entropy becomes an arbitrary function of the gravitational radius.

$R \to r_+$

$r_- \to r_+$

$R \to r_+ \quad r_- \to r_+$

Case	Pressure p	Rotational Velocity v	Temperature T	Entropy S
1	∞	0	∞	$\frac{A_+}{4G}$
2	Finite	1	0	Arbitrary $S(r_+)$
3	Finite	< 1	Finite	$\frac{A_+}{4G}$

Thank you.