Superradiant amplification by stars and black-holes

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Overview

Outline

Motivation

In this thesis we study a phenomenon of amplification of radiation, called *superradiance*, in astrophysical objects.

Areas of impact: Astrophysics, gravitation and particle physics.

Some applications:

- Search of dark matter candidates and physics beyond the Standard Model (Arvanitaki et al. 2011),
- Constrain the mass of ultralight degrees of freedom such as the photon and the graviton (Pani et al. 2012, Brito et al. 2013),
- Study the existence of hairy black-hole and star solutions (Herdeiro et al. 2014).

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A radiation enhancement process where the scattering of incident waves on a **rotating** and **dissipative** system results in reflected waves with larger amplitude.

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Superradiance condition: $$\omega < m\Omega \tag{2}$$

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Confinement of superradiant modes $\rightarrow$ **Instabilities**

Presence of a **massive** field: mass works as a natural confinement.

Outline

Wave equation

Inside metric $(r < R)$ (Shapiro and Teukolsky 1983)

$$ds^2 = -e^{2\varphi}dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (3)$$

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$$m(r) = \frac{4}{3}\pi r^3 \rho, \qquad e^{\varphi} = \frac{3}{2}\sqrt{1 - \frac{2M}{R}} - \frac{1}{2}\sqrt{1 - \frac{2Mr^2}{R^3}}.$$

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Stress-energy tensor $(r < R)$

$$T^{ab} = (\rho + P)\, U^a U^b + P g^{ab}. \quad (5)$$

Wave equation

Since $U^a = \left(\sqrt{-g^{tt}}, 0, 0, 0\right)$,

$$P = \rho \left(\frac{\sqrt{1 - 2Mr^2/R^3} - \sqrt{1 - 2M/R}}{3\sqrt{1 - 2M/R} - \sqrt{1 - 2Mr^2/R^3}} \right), \qquad \rho = \frac{3M}{4\pi R^3}. \tag{6}$$

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Klein-Gordon equation $(r > R \Rightarrow \alpha = 0)$:

$$\nabla_a \nabla^a \Psi + \alpha \frac{\partial \Psi}{\partial t} = \mu^2 \Psi. \tag{7}$$

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Co-rotating frame ($r < R$ only):

$$\phi' = \phi - \Omega t \quad \implies \quad \omega' = \omega - m\Omega. \tag{8}$$

Wave equation

Separation of variables with Teukolsky's ansatz

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Angular equation

$$-\cot\theta \frac{\partial_\theta S}{S} + \frac{m^2}{\sin^2\theta} - \frac{\partial_\theta \partial_\theta S}{S} = \lambda. \tag{10}$$

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Simplifying

$$\frac{\sin\theta}{S}\partial_\theta\left(\sin\theta\,\partial_\theta S\right) + \lambda\sin^2\theta = m^2 \quad \Longrightarrow \quad \lambda = l\,(l+1)\,. \tag{11}$$

$$S(\theta) = \sqrt{\frac{(2l+1)\,(l-m)!}{4\pi\,(l+m)!}}\mathcal{P}_l^m(\cos\theta)\,. \tag{12}$$

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Interior coefficients:

$$A_r(r) = e^{-\varphi} e^{\varphi'} + \frac{2}{r} + \left(1 - \frac{2m(r)}{r}\right)^{-1} \left(\frac{m(r)}{r^2} - \frac{m'(r)}{r}\right), \tag{13}$$

$$B_r(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1} \left(\omega^2 e^{-2\varphi} - \mu^2 - i\omega\alpha - \frac{\lambda}{r^2}\right). \tag{14}$$

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Exterior coefficients:

$$A_r(r) = \frac{2}{r}\left(\frac{r-M}{r-2M}\right), \tag{15}$$

$$B_r(r) = \left(1 - \frac{2M}{r}\right)^{-1} \left[\omega^2 \left(1 - \frac{2M}{r}\right)^{-1} - \mu^2 - \frac{\lambda}{r^2}\right] \tag{16}$$

Wave equation

Coordinate and function transformation $(r > R)$

$$u(r) = rR(r), \qquad \frac{dr}{dr^*} = \left(1 - \frac{2M}{r}\right), \tag{17}$$

$$\frac{d^2u}{dr^{*2}} + \left[\omega^2 - V(r)\right] u = 0. \tag{18}$$

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Limit $V(\infty) = \mu^2 \implies u(\infty) = A_\infty e^{+i\sqrt{\omega^2-\mu^2}r^*} + B_\infty e^{-i\sqrt{\omega^2-\mu^2}r^*}$.

$$R_\infty(r) = e^{\pm i\sqrt{\omega^2-\mu^2}r} r^{\pm\beta} \sum_n D_n \frac{1}{r^n}, \qquad \beta = i\frac{M\left(2\omega^2 - \mu^2\right)}{\sqrt{\omega^2 - \mu^2}}. \tag{19}$$

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Limit $r \to 0$

$$R''(r) + \frac{2}{r}R'(r) - \frac{l(l+1)}{r^2}R(r) = 0. \tag{20}$$

$$R_0(r) = r^l \sum_n C_n r^{2n}. \tag{21}$$

Outline

Amplification

Series expansions: C_n and D_n as functions of C_0 and D_1.

- C_0: **arbitrary** (we choose $C_0 = 1$),
- D_1: A_∞ for the **outgoing** wave; B_∞ for the **ingoing** wave.

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Reflection coefficient

$$Z(\omega) = \frac{|A_\infty|^2}{|B_\infty|^2}. \tag{22}$$

- $Z > 1 \implies$ medium **amplifies**,
- $Z < 1 \implies$ medium **absorbs**.

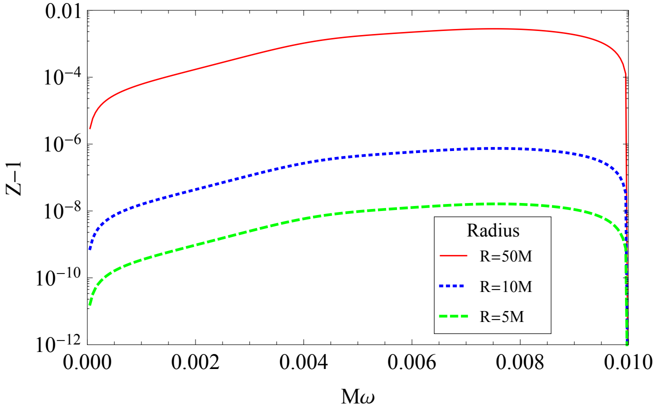

Figure: $\mu = 0$, $\alpha M = 0.1$, $\Omega M = 0.01$, $l = m = 1$.

Outline

Stability

If $\mu > \omega \implies \pm i\sqrt{\omega^2 - \mu^2} = \mp\sqrt{\mu^2 - \omega^2}$

$$u(r) = A_\infty e^{-\sqrt{\mu^2-\omega^2}r^*} + B_\infty e^{+\sqrt{\mu^2-\omega^2}r^*}, \tag{23}$$

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Regularity: $B_\infty = 0 \implies \omega = \omega_R + i\omega_I$ **Quasi-boundstates**

$$\Psi = e^{\omega_I t} e^{-im\phi - i\omega_R t} R\left(r\right) S\left(\theta\right). \tag{24}$$

- $\omega_I > 0 \implies$ system is **unstable**,
- $\omega_I < 0 \implies$ system is **stable**.

Stability

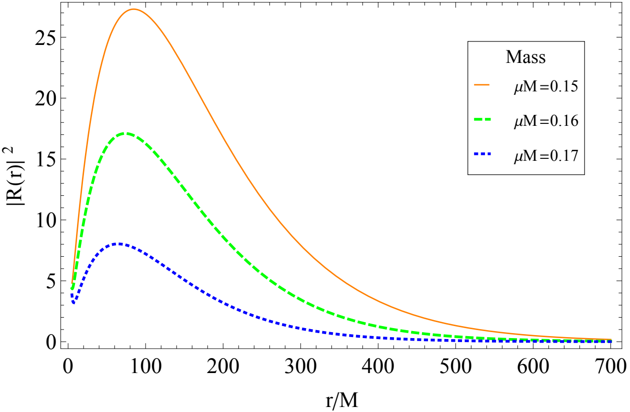

Figure: $\Omega M = 0.1$, $\alpha M = 5$, $R = 5M$, $l = m = 1$.

Stability

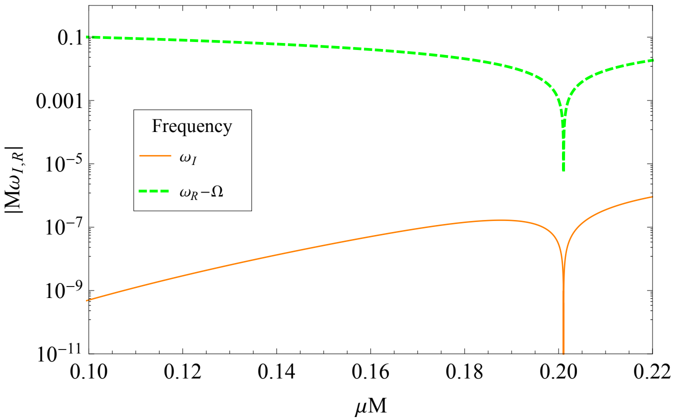

Figure: $\alpha M = 20$, $R = 4M$, $\Omega M = 0.2$, $l = m = 1$.

Newtonian limit

If $M/R << 1$ and $\Omega R << 1$

$$R''(r) + \frac{2}{r}R'(r) + \left(\omega^2 - i\omega\alpha - \frac{\lambda}{r^2}\right)R(r) = 0. \tag{25}$$

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Solution:

$$R_{int}(r) = j_l\left(r\sqrt{(\omega - m\Omega)(i\alpha + \omega - m\Omega)}\right), \quad r < R; \tag{26}$$

$$R_{ext}(r) = Aj_l(r\omega) + By_l(r\omega), \quad r > R. \tag{27}$$

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$$Z = 1 + \frac{4\alpha R^2(\Omega - \omega)(\omega R)^{2l+1}}{(2l+1)!!(2l+3)!!}, \tag{28}$$

$$\alpha = \frac{1}{M}, R = 2M, l = 1 \implies Z = 1 + \frac{16}{45}M(\Omega - \omega)(2M\omega)^3. \tag{29}$$

Newtonian limit

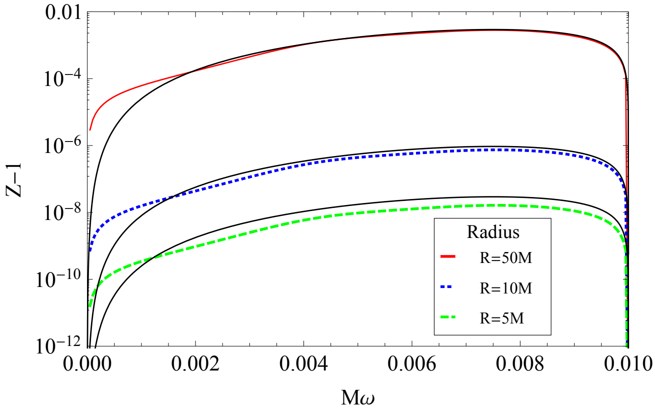

Figure: Comparison of the reflection coefficients obtained numerically with the analytical results.

Conclusions

- Stars display superradiance when dissipation is properly included;
- There are no unstable modes for non-rotating stars;
- There are no unstable modes for massless perturbations;
- Unstable modes only occur in the superradiant regime;
- Newtonian systems also display superradiance;
- Relativistic effects related to frame-dragging are neglectable;
- More sofisticated models are needed to describe dissipation.

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Relativistic effects

New metric:

$$ds^2 = -e^{2\varphi} dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 \left[d\theta^2 + \sin^2\theta \left(d\phi - \zeta(r)\, dt\right)^2\right]. \tag{30}$$

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New quadrivelocity: $U^a =$
$\left(\left[-\left(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right)\right]^{-\frac{1}{2}}, 0, 0, \Omega\left[-\left(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right)\right]^{-\frac{1}{2}}\right)$

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Interior equation:

$$\zeta''(r) + \frac{1}{2}\left(\frac{8}{r} - \frac{B'(r)}{B(r)} - \frac{f'(r)}{f(r)}\right)\zeta'(r) = 16\pi(\rho + P)(\zeta(r) - \Omega)B(r). \tag{31}$$

Exterior equation:

$$\zeta''(r) + \frac{4}{r}\zeta'(r) = 0, \tag{32}$$

Relativistic effects

Outside solution:

$$\zeta_{out}(r) = \frac{2J}{r^3}, \tag{33}$$

Boundary condition:

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$$\bar{\zeta}(r) = \Omega - \zeta_{int}(r) \implies J = \frac{1}{6}R^4\bar{\zeta}'(R), \qquad \Omega = \bar{\zeta}(R) + \frac{2J}{R^3}. \tag{35}$$

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New coefficient:

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Relativistic effects

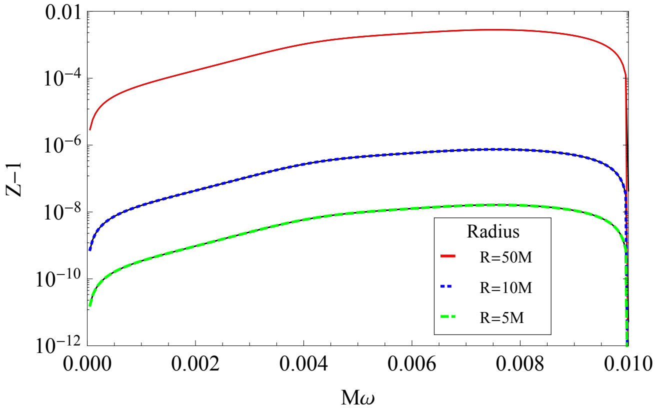

Figure: Comparison of the reflection coefficients obtained numerically with the relativistic results