Gravitomagnetism and the meaning of the scalar invariants of the Riemann tensor

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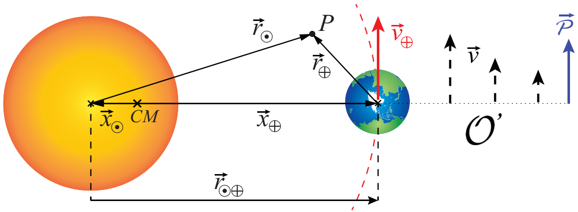
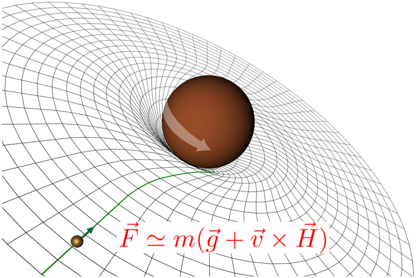
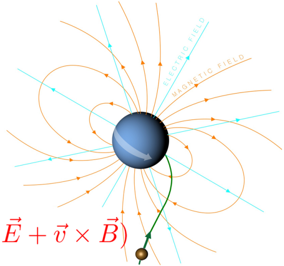


Introduction



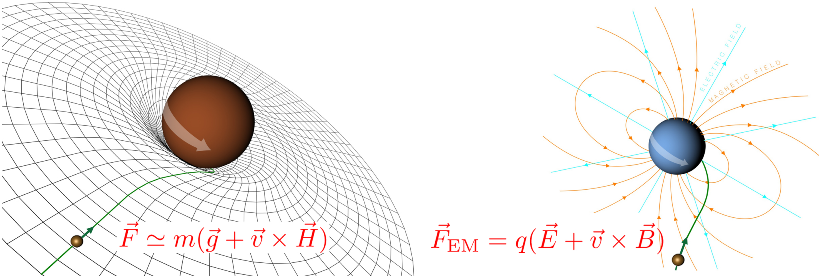
$$\vec{F} \simeq m(\vec{g} + \vec{v} \times \vec{H})$$



$$\vec{F}_{\mathrm{EM}} = q(\vec{E} + \vec{v} \times \vec{B})$$

- Mass/energy currents originate effects strongly resembling magnetism: “gravitomagnetism” (Ex: frame dragging)
- Translational gravitomagnetism (detected to high precision)
  - Orbital perturbations: Binary Pulsars; Moon’s orbit
  - Spin vector geodetic precession: Gravity Probe B, binary pulsars
- Rotational gravitomagnetism (more elusive): Earth’s, detected by Laser Ranging to the LAGEOS satellites, and with Gravity Probe B

Introduction



- Translational gravitomagnetism is dubbed "extrinsic", rotational gravitomagnetism dubbed "intrinsic",
- Classification based on the formal analogy between the quadratic invariants of the Riemann and Maxwell tensors

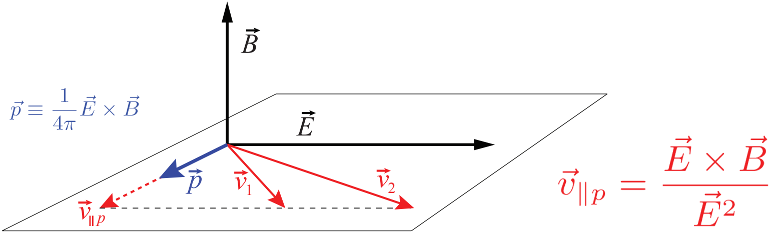
$$\{\mathbf{F} \cdot \mathbf{F},\ \star\mathbf{F} \cdot \mathbf{F}\} \leftrightarrow \{\mathbf{R} \cdot \mathbf{R},\ \star\mathbf{R} \cdot \mathbf{R}\}$$

- Gravitomagnetic effects measured in regions where $\star\mathbf{R} \cdot \mathbf{R} = 0$ implied to be different in nature from the ones where $\star\mathbf{R} \cdot \mathbf{R} \neq 0$

Scalar Invariants – Electromagnetism

$$\vec{E}^2 - \vec{B}^2 = -\frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \qquad \vec{E} \cdot \vec{B} = -\frac{1}{4} F_{\alpha\beta} \star F^{\alpha\beta}$$

- $\vec{E} \cdot \vec{B} \neq 0 \Rightarrow \vec{E}$ and $\vec{B}$ are both non-vanishing for all observers
- $\vec{E} \cdot \vec{B} = 0$ and $\vec{E}^2 - \vec{B}^2 > 0 \Rightarrow$ there are observers for which $\vec{B}$ vanishes.



- their velocity is of the form $\vec{v} = \vec{v}_{\|p} + \vec{v}_{\|E}$, having a component $\vec{v}_{\|p}$ along the Poynting vector, and an *arbitrary* component $\vec{v}_{\|E}$ along the electric field.

Scalar Invariants – Electromagnetism

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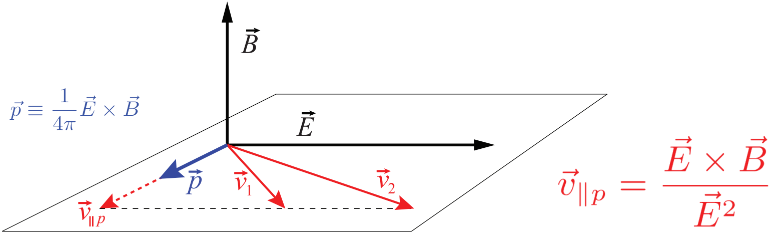

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- $\vec{E} \cdot \vec{B} = 0$ and $\vec{E}^2 - \vec{B}^2 < 0 \Rightarrow$ there are observers for which the electric field $\vec{E}$ vanishes (analogously).

Scalar Invariants – Electromagnetism

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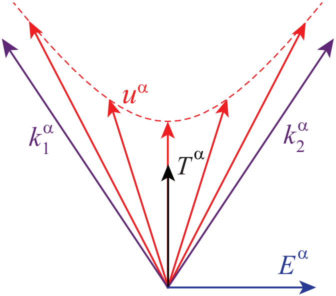

- $F_{\alpha\beta}$ has two principal null directions k_1^α and k_2^α, such that $k^{[\alpha} F^{\beta]}{}_\gamma k^\gamma = 0$
- the 4-velocities u^α of the observers measuring $B^\alpha = 0$ are any unit time-like vector lying in the plane spanned by k_1^α and k_2^α.

Scalar Invariants – Electromagnetism

$$\vec{E}^2 - \vec{B}^2 = -\frac{1}{2}F_{\alpha\beta}F^{\alpha\beta} \qquad \vec{E}\cdot\vec{B} = -\frac{1}{4}F_{\alpha\beta}\star F^{\alpha\beta}$$

- $\vec{E}\cdot\vec{B} = 0$ and $\vec{E}^2 = \vec{B}^2 = 0 \;\Rightarrow$ null field: either $F^{\alpha\beta} = 0$, or pure radiation.
- $\vec{E}^2 - \vec{B}^2$ and $\vec{E}\cdot\vec{B}$ are the only algebraically independent invariants one can define from the Maxwell tensor $F^{\alpha\beta}$.

Curvature scalar invariants (vacuum)

In vacuum, the Riemann tensor decomposes irreducibly as

$$
\begin{aligned}
R_{\alpha\beta}{}^{\gamma\delta} &= 4\left\{2U_{[\alpha}U^{[\gamma} + g_{[\alpha}{}^{[\gamma}\right\}\mathbb{E}_{\beta]}{}^{\delta]} \\
&\quad +2\left\{\epsilon_{\alpha\beta\mu\nu}\mathbb{H}^{\mu[\delta}U^{\gamma]}U^{\nu} + \epsilon^{\gamma\delta\mu\nu}\mathbb{H}_{\mu[\beta}U_{\alpha]}U_{\nu}\right\}
\end{aligned}
$$

- $\mathbb{E}_{\alpha\beta} = R_{\alpha\mu\beta\nu}U^{\mu}U^{\nu} \equiv$ electric part of the Riemann tensor (gravitoelectric tidal tensor)
- $\mathbb{H}_{\alpha\beta} = \star R_{\alpha\mu\beta\nu}U^{\mu}U^{\nu} \equiv$ magnetic part of the Riemann tensor (gravitomagnetic tidal tensor)

- Analogous to the splitting of $F_{\alpha\beta}$ into electric $E^{\alpha} \equiv F^{\alpha\beta}U_{\beta}$ and magnetic fields $E^{\alpha} \equiv \star F^{\alpha\beta}U_{\beta}$

$$F_{\alpha\beta} = 2U_{[\alpha}E_{\beta]} + \epsilon_{\alpha\beta\gamma\delta}B^{\gamma}U^{\delta}$$

Curvature scalar invariants (vacuum)

In vacuum, one can construct 4 independent scalar invariants from the Riemann tensor (would be 14 in general).

- The two quadratic invariants

$$\begin{aligned} \mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} &= \frac{1}{8} R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \equiv \frac{1}{8} \mathbf{R} \cdot \mathbf{R} \\ \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} &= \frac{1}{16} R_{\alpha\beta\gamma\delta} \star R^{\alpha\beta\gamma\delta} \equiv \frac{1}{16} \star\mathbf{R} \cdot \mathbf{R} \end{aligned}$$

formally analogous to the electromagnetic invariants

$$\begin{aligned} E^{\alpha}E_{\alpha} - B^{\alpha}B_{\alpha} &= -\tfrac{1}{2} F^{\alpha\beta}F_{\alpha\beta} \equiv \frac{1}{2} \mathbf{F} \cdot \mathbf{F} \\ E^{\alpha}B_{\alpha} &= -\tfrac{1}{4} \star F^{\alpha\beta}F_{\alpha\beta} \equiv -\frac{1}{4} \star \mathbf{F} \cdot \mathbf{F} \end{aligned}$$

Curvature scalar invariants (vacuum)

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- and the two cubic invariants

$$\begin{aligned} \mathbb{E}^{\alpha}{}_{\beta}\mathbb{E}^{\beta}{}_{\gamma}\mathbb{E}^{\gamma}{}_{\alpha} - 3\mathbb{E}^{\alpha}{}_{\beta}\mathbb{H}^{\beta}{}_{\gamma}\mathbb{H}^{\gamma}{}_{\alpha} &= \frac{1}{16} R^{\alpha\beta}{}_{\lambda\mu} R^{\lambda\mu}{}_{\rho\sigma} R^{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} \\ \mathbb{H}^{\alpha}{}_{\beta}\mathbb{H}^{\beta}{}_{\gamma}\mathbb{H}^{\gamma}{}_{\alpha} - 3\mathbb{E}^{\alpha}{}_{\beta}\mathbb{E}^{\beta}{}_{\gamma}\mathbb{H}^{\gamma}{}_{\alpha} &= \frac{1}{16} R^{\alpha\beta}{}_{\lambda\mu} R^{\lambda\mu}{}_{\rho\sigma} R^{\rho\sigma} \star R^{\rho\sigma}{}_{\alpha\beta} \end{aligned}$$

- These invariants are related to conditions for the existence of observers for which $\mathbb{E}_{\alpha\beta}$ or $\mathbb{H}_{\alpha\beta}$ vanish
  - all are needed, and still they are not sufficient

Curvature scalar invariants (vacuum)

Define $\mathbb{Q}_{\alpha\beta} = \mathbb{E}_{\alpha\beta} - i\mathbb{H}_{\alpha\beta}$

- Sum of two spatial tensors, each of them diagonalizable
- The existence of observers for which $\mathbb{H}_{\alpha\beta} = 0$ ($\mathbb{E}_{\alpha\beta} = 0$) is equivalent to existence of observers for which
  - $\mathbb{Q}_{\alpha\beta}$ is diagonalizable,
  - *with* real (purely imaginary) eigenvalues
  - *and* allowing for a basis of real orthonormal eigenvectors
- these are observer independent features — the eigenvalue problem for $\mathbb{Q}_{\alpha\beta}$ is a way of formulating the Petrov classification
- Diagonalizable types are Petrov types I and D

Curvature scalar invariants (vacuum)

Existence of observers measuring $\mathbb{H}_{\alpha\beta} = 0$ (or $\mathbb{E}_{\alpha\beta} = 0$) needs

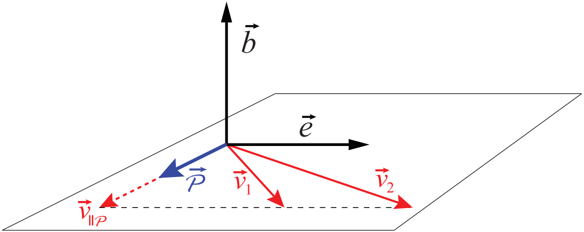
- $\star\mathbf{R} \cdot \mathbf{R} = 0$, $\mathbf{R} \cdot \mathbf{R} > 0$ (< 0)
- $\mathbb{A} = \mathbb{B} = 0$ or $\mathbb{M} > 0$ (real) or Petrov type D

$$
\begin{aligned}
\mathbb{M} &\equiv I^3/(\mathbb{A} + i\mathbb{B})^2 - 6 \\
\mathbb{A} &\equiv \frac{1}{16} R^{\alpha\beta}{}_{\lambda\mu} R^{\lambda\mu}{}_{\rho\sigma} R^{\rho\sigma} R^{\rho\sigma}{}_{\alpha\beta} \\
\mathbb{B} &\equiv \frac{1}{16} R^{\alpha\beta}{}_{\lambda\mu} R^{\lambda\mu}{}_{\rho\sigma} R^{\rho\sigma} \star R^{\rho\sigma}{}_{\alpha\beta} \\
I &\equiv \frac{1}{8} \mathbf{R} \cdot \mathbf{R} - \frac{i}{8} \star \mathbf{R} \cdot \mathbf{R}
\end{aligned}
$$

- the Petrov type D condition cannot be stated in terms of invariants
  - as $\mathbb{M} = 0$ for both type D and the *non-diagonalizable* type II

Vacuum Petrov type D — strong electromagnetic analogy

- $\star\mathbf{R}\cdot\mathbf{R}\neq 0 \Rightarrow \mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are both non-vanishing for all observers.
- $\star\mathbf{R}\cdot\mathbf{R}=0$ and $\mathbf{R}\cdot\mathbf{R}>0\ (<0) \Rightarrow$ there are observers for which $\mathbb{H}_{\alpha\beta}$ $(\mathbb{E}_{\alpha\beta})$ vanishes



- they are boosted along the "super-Poynting" vector $\vec{\mathcal{P}}=\overleftrightarrow{\mathbb{E}}\times\overleftrightarrow{\mathbb{H}}$ with a velocity

$$\vec{v}_{\parallel\mathcal{P}}=\frac{\overleftrightarrow{\mathbb{E}}\times\overleftrightarrow{\mathbb{H}}}{9|\lambda|^2A(A+1)}\quad \text{i.e.}\quad v^{\alpha}_{\parallel\mathcal{P}}=\frac{\epsilon^{\alpha\beta\gamma\delta}\mathbb{E}_{\beta\mu}\mathbb{H}^{\mu}{}_{\gamma}u_{\delta}}{9|\lambda|^2A(A+1)}$$

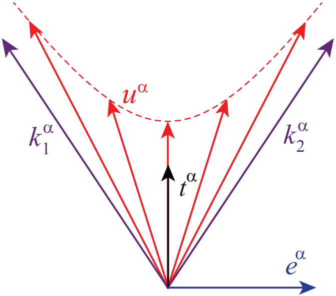
- analogous to the electromagnetic counterpart

$$\vec{v}_{\parallel p}=\frac{\vec{E}\times\vec{B}}{\vec{E}^2}\quad \text{i.e.}\quad v^{\alpha}_{\parallel p}=\frac{\epsilon^{\alpha}{}_{\sigma\tau\beta}E^{\sigma}B^{\tau}u^{\beta}}{E_{\nu}E^{\nu}}$$

Vacuum Petrov type D — strong electromagnetic analogy

- $\star\mathbf{R}\cdot\mathbf{R}\neq 0 \Rightarrow \mathbb{E}_{\alpha\beta}$ and $\mathbb{H}_{\alpha\beta}$ are both non-vanishing for all observers.

- $\star\mathbf{R}\cdot\mathbf{R}=0$ and $\star\mathbf{R}\cdot\mathbf{R}>0$ (<0) $\Rightarrow$ there are observers for which $\mathbb{H}_{\alpha\beta}$ $(\mathbb{E}_{\alpha\beta})$ vanishes



- $R_{\alpha\beta\gamma\delta}$ has two principal null directions k_1^α and k_2^α, such that $k^{[\alpha}R^{\beta]}{}_{\gamma\delta\epsilon}k^\gamma k^\epsilon = 0$
- the 4-velocities u^α of the observers measuring $\mathbb{H}_{\alpha\beta}=0$ $(\mathbb{E}_{\alpha\beta}=0)$ are any unit time-like vector lying in the plane spanned by k_1^α and k_2^α.

Vacuum Petrov type I

In the (vacuum) Petrov type I case the situation is different, and not analogous to electromagnetism:

- the observer measuring $\mathbb{H}_{\alpha\beta} = 0$ ($\mathbb{E}_{\alpha\beta} = 0$) (when $\star\mathbf{R}\cdot\mathbf{R} = 0$ and $\star\mathbf{R}\cdot\mathbf{R} > 0$ (< 0) are satisfied) is *unique*
- they are not obtained by boosting in the direction of the super-Poynting vector $\vec{\mathcal{P}}$

General case in the presence of sources

- a similar classification always holds for the Weyl tensor **C**, and its electric $\mathcal{E}_{\alpha\beta} \equiv C_{\alpha\mu\beta\nu} U^\mu U^\nu$ and magnetic $\mathcal{H}_{\alpha\beta} \equiv \star C_{\alpha\mu\beta\nu} U^\mu U^\nu$ parts

- for the Riemann tensor, the conditions for the existence of observers for which $\mathbb{H}_{\alpha\beta} = 0$ and $\mathbb{E}_{\alpha\beta} = 0$ are not known.

- Riemann invariants do not involve only $\mathbb{E}_{\alpha\beta} \equiv R_{\alpha\mu\beta\nu} U^\mu U^\nu$ and $\mathbb{H}_{\alpha\beta} \equiv \star R_{\alpha\mu\beta\nu} U^\mu U^\nu$, but also a third spatial tensor $\mathbb{F}_{\alpha\beta} \equiv \star R \star_{\alpha\mu\beta\nu} U^\mu U^\nu$

- We only know that $\star \mathbf{R} \cdot \mathbf{R} \neq 0$ implies $\mathbb{H}_{\alpha\beta}$ for all observers since
  - $\star \mathbf{R} \cdot \mathbf{R} = \star \mathbf{C} \cdot \mathbf{C}$
  - hence $\star \mathbf{C} \cdot \mathbf{C} \neq 0 \Rightarrow \mathcal{H}_{\alpha\beta} \neq 0 \Rightarrow \mathbb{H}_{\alpha\beta} \neq 0$ since $\mathcal{H}_{\alpha\beta} = \mathbb{H}_{[\alpha\beta]}$

Origin of the Invariant structure

"spacetime geometry and the corresponding curvature invariants are affected and determined, not only by mass-energy, but also by mass-energy currents relative to other mass, that is, mass-energy currents not generable nor eliminable by any Lorentz transformation" (Ciufolini-Wheeler 1995)

- But there is no (unambiguous) way of determining relative motion of distant bodies in a curved spacetime (no global notion of parallelism)
  - relative motion well defined only when observers/bodies are at the same point
- Definitions of relative velocity have been proposed in the literature; but a direct relation with the curvature invariants seems to be ruled out (notion of relative rest non-transitive, and most non-symmetric)
- We need to do better...

Maxwell equations	Differential Bianchi Identities
$\nabla^{\perp} \times \vec{B} = \dot{\vec{E}} + 4\pi\vec{j} - \vec{a} \times \vec{B} - \sigma^{\hat{i}\hat{j}} E_{\hat{j}} \vec{e}_{\hat{i}} + \frac{2}{3}\theta\vec{E}$	$\mathrm{curl}\mathcal{H}_{\hat{i}\hat{j}} = \dot{\mathcal{E}}_{\hat{i}\hat{j}} + 4\pi\left[(\rho+p)\sigma_{\hat{i}\hat{j}} + \nabla^{\perp}_{\langle\hat{i}} J_{\hat{j}\rangle} + \dot{\pi}_{\hat{i}\hat{j}}\right]$ + contractions of $\{\mathcal{E}_{\hat{i}\hat{j}}, \mathcal{H}_{\hat{i}\hat{j}}, J^{\hat{i}}, \pi_{\hat{i}\hat{j}}\}$ with $\{a_{\hat{i}}, \omega_{\hat{i}}, \sigma_{\hat{i}\hat{j}}, \theta\}$
$\nabla^{\perp} \cdot \vec{B} = -2\vec{\omega} \cdot \vec{E}$	$\nabla^{\perp}_{\hat{j}} \mathcal{H}^{\hat{j}}{}_{\hat{i}} = -4\pi\left[2(\rho+p)\omega_{\hat{i}} + (\nabla^{\perp} \times \vec{J})_{\hat{i}}\right]$ + contract. of $\{\mathcal{E}_{\hat{i}\hat{j}}, \pi_{\hat{i}\hat{j}}\}$ with $\{\omega_{\hat{i}}, \sigma_{\hat{i}\hat{j}}\}$
• Inertial frame	• Post-Newtonian regime
$\nabla \times \vec{B} = \dot{\vec{E}} + 4\pi\vec{j}$	$\mathrm{curl}\mathcal{H}_{ij} = \dot{\mathcal{E}}_{ij} + 4\pi J_{\langle i,j\rangle}$
$\nabla \cdot \vec{B} = 0$	$\mathcal{H}^{j}{}_{i,j} = -4\pi(\nabla \times \vec{J})_i$

- If an inertial frame exists where $\dot{\vec{E}} + 4\pi\vec{j} = \vec{0}$ everywhere, then $\vec{B} = 0$ globally in that frame $\Rightarrow \vec{E} \cdot \vec{B} = 0$ everywhere.
  - Ex: system of point charges; if they are all at rest, $\vec{B} = 0$.
  - Converse is not true.
- In *flat spacetime*, indeed one can relate the vanishing of $\vec{E} \cdot \vec{B}$ with an absence relative motion between the sources

Maxwell equations	Differential Bianchi Identities
$\nabla^\perp \times \vec{B} = \dot{\vec{E}} + 4\pi\vec{j} - \vec{a} \times \vec{B} - \sigma^{\hat{i}\hat{j}} E_{\hat{j}} \vec{e}_{\hat{i}} + \frac{2}{3}\theta\vec{E}$	$\mathrm{curl}\mathcal{H}_{\hat{i}\hat{j}} = \dot{\mathcal{E}}_{\hat{i}\hat{j}} + 4\pi\left[(\rho + p)\sigma_{\hat{i}\hat{j}} + \nabla^\perp_{\langle\hat{i}} J_{\hat{j}\rangle} + \dot{\pi}_{\hat{i}\hat{j}}\right]$ + contractions of $\{\mathcal{E}_{\hat{i}\hat{j}}, \mathcal{H}_{\hat{i}\hat{j}}, J^{\hat{i}}, \pi_{\hat{i}\hat{j}}\}$ with $\{a_{\hat{i}}, \omega_{\hat{i}}, \sigma_{\hat{i}\hat{j}}, \theta\}$
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$\nabla \times \vec{B} = \dot{\vec{E}} + 4\pi\vec{j}$	$\mathrm{curl}\mathcal{H}_{ij} = \dot{\mathcal{E}}_{ij} + 4\pi J_{\langle i,j\rangle}$
$\nabla \cdot \vec{B} = 0$	$\mathcal{H}^j{}_{i,j} = -4\pi(\nabla \times \vec{J})_i$

- If an inertial frame exists where $\dot{\vec{E}} + 4\pi\vec{j} = \vec{0}$ everywhere, then $\vec{B} = 0$ globally in that frame $\Rightarrow \vec{E} \cdot \vec{B} = 0$ everywhere.
- implication above is guaranteed only for inertial frames; in other frames ω^α, $\sigma_{\alpha\beta}$, θ contribute as sources for $\vec{B}$.
  - Ex: system of charges in rigid rotational motion; they are at rest in the co-rotating frame, yet $\vec{B} \neq 0$ and $\vec{B} \cdot \vec{E} \neq 0$.

Maxwell equations	Differential Bianchi Identities
$\nabla^\perp \times \vec{B} = \dot{\vec{E}} + 4\pi\vec{j}$ $-\vec{a}\times\vec{B} - \sigma^{\hat{i}\hat{j}}E_{\hat{j}}\vec{e}_{\hat{i}} + \frac{2}{3}\theta\vec{E}$	$\mathrm{curl}\mathcal{H}_{\hat{i}\hat{j}} = \dot{\mathcal{E}}_{\hat{i}\hat{j}} + 4\pi\left[(\rho+p)\sigma_{\hat{i}\hat{j}} + \nabla^\perp_{\langle\hat{i}}J_{\hat{j}\rangle} + \dot{\pi}_{\hat{i}\hat{j}}\right]$ $+ \text{ contractions of } \{\mathcal{E}_{\hat{i}\hat{j}}, \mathcal{H}_{\hat{i}\hat{j}}, J^{\hat{i}}, \pi_{\hat{i}\hat{j}}\}$ $\text{with } \{a_{\hat{i}}, \omega_{\hat{i}}, \sigma_{\hat{i}\hat{j}}, \theta\}$
$\nabla^\perp \cdot \vec{B} = -2\vec{\omega}\cdot\vec{E}$	$\nabla^\perp_{\hat{j}}\mathcal{H}^{\hat{j}}{}_{\hat{i}} = -4\pi\left[2(\rho+p)\omega_{\hat{i}} + (\nabla^\perp\times\vec{J})_{\hat{i}}\right]$ $+ \text{ contract. of } \{\mathcal{E}_{\hat{i}\hat{j}}, \pi_{\hat{i}\hat{j}}\} \text{ with } \{\omega_{\hat{i}}, \sigma_{\hat{i}\hat{j}}\}$
• Inertial frame	• Post-Newtonian regime
$\nabla \times \vec{B} = \dot{\vec{E}} + 4\pi\vec{j}$	$\mathrm{curl}\mathcal{H}_{ij} = \dot{\mathcal{E}}_{ij} + 4\pi J_{\langle i,j\rangle}$
$\nabla\cdot\vec{B} = 0$	$\mathcal{H}^j{}_{i,j} = -4\pi(\nabla\times\vec{J})_i$

- Similar statements for gravity, replacing:
  - B^α by $\mathcal{H}_{\alpha\beta}/\mathbb{H}_{\alpha\beta}$
  - $\star\mathbf{F}\cdot\mathbf{F}$ by $\star\mathbf{R}\cdot\mathbf{R}$
  - inertial frames by Post-Newtonian frames
- Relation between $\star\mathbf{R}\cdot\mathbf{R}$ and the relative motion of the sources recovered at 1PN order (and PN frames)

Point charge vs Schwarzschild solution

Point charge: $$\begin{cases} \vec{E}^2 - \vec{B}^2 = \dfrac{q^2}{r^4} > 0 \\ \vec{E} \cdot \vec{B} = 0 \quad \text{(everywhere)} \end{cases}$$

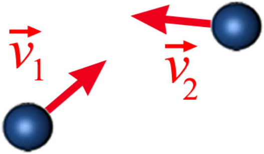
$\Rightarrow$Everywhere there is as class of observers for which $\vec{B} = 0$

- static observers, and observers moving radially (as the component $\vec{v}_{||E}$ along $\vec{E}$ is arbitrary)

Schwarzschild (Petrov type D): $$\begin{cases} \mathbb{E}^{\alpha\gamma}\mathbb{E}_{\alpha\gamma} - \mathbb{H}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \dfrac{6m^2}{r^6} > 0 \\ \mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = 0 \text{ (everywhere)} \end{cases}$$

- Everywhere there is as class of observers for which $\mathbb{H}_{\alpha\gamma} = 0$ $\Rightarrow$ static observers (outside the horizon), and observers moving radially, just like in electromagnetism

Two charges

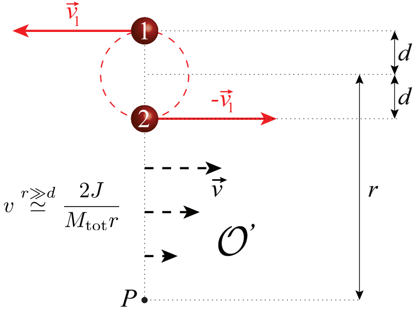


$$\vec{E}\cdot\vec{B} \simeq \frac{Q_1 Q_2}{r_1^3 r_2^3}\left[(\vec{v}_1\times\vec{r}_1)\cdot\vec{r}_2 + (\vec{v}_2\times\vec{r}_2)\cdot\vec{r}_1\right] \quad (\neq 0 \text{ generically})$$

- Generically, the magnetic field does not vanish for any observer
  - consistent with the fact that there is no inertial frame where *both* charges are at rest
- but if the motion is coplanar, $\vec{E}\cdot\vec{B} = 0$ in the plane of the motion
  - at every point in the plane there are observers for which $\vec{B} = 0$

circular motion; $\vec{v}_1 = -\vec{v}_2$

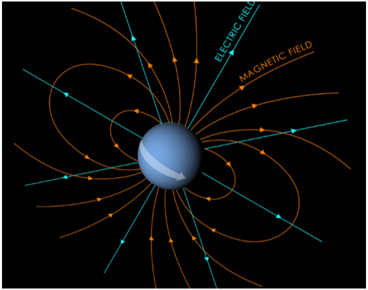
Two charges —



$$\begin{cases} \vec{E} \cdot \vec{B} = 0 \text{ in the plane of the motion} \\ \vec{E} \cdot \vec{B} \neq 0 \quad \text{elsewhere} \end{cases}$$

- For the static observer $\mathcal{O}$ at P, $\vec{B} = \vec{B}_1 + \vec{B}_2 \neq 0$ because $|\vec{B}_2| > |\vec{B}_1|$ (since particle 2 is closer to P)
- By moving with 3-velocity $\vec{v}$ in the same direction as particle 2, observer $\mathcal{O}'$ decreases its relative velocity to particle 2, and increases its relative velocity to particle 1
- That means decreasing $|\vec{B'}_2|$ whilst increasing $|\vec{B}'_1|$; they cancel out ($\vec{B}' = 0$) for $v \simeq 2vd/r$

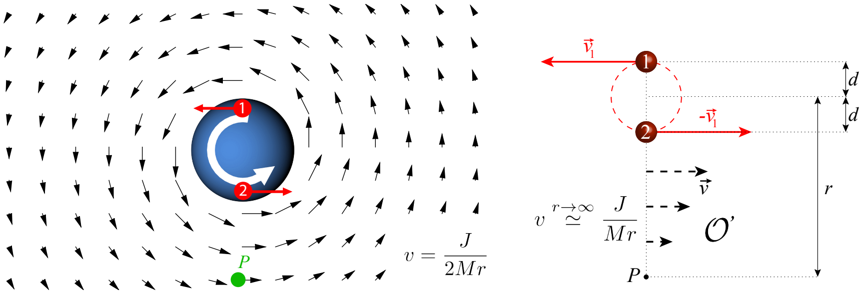
Spinning Charge



$$\begin{cases} \vec{E}^2 - \vec{B}^2 = \dfrac{q^2}{r^4} - \dfrac{\mu^2(5 + 3\cos 2\theta)}{2r^6} > 0 \\ \vec{E} \cdot \vec{B} = \dfrac{2\mu q \cos\theta}{r^5} \end{cases}$$

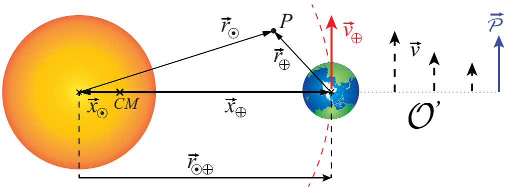
- $\vec{E} \cdot \vec{B} = 0$ in the equatorial plane, $\neq 0$ elsewhere
- In the equatorial plane there are observers for which $\vec{B}' = 0$;
- $\vec{B}' \neq 0$ elsewhere for *every* observer

Spinning Charge



- $\vec{E} \cdot \vec{B} = 0$ in the equatorial plane, $\neq 0$ elsewhere
- In the equatorial plane there are observers for which $\vec{B}' = 0$;
- $\vec{B}' \neq 0$ elsewhere for *every* observer
- explained by the same reasoning as the system of two charges (rotating body may the cast as an assembly of pairs of elements in antipodal positions)
- the velocities asymptotically match up to a factor of 2
- congruence of observers for which $\vec{B}' = 0$ is a shearing one (besides rotating)

Gravitational 2 body system: Earth $\equiv \oplus$; Sun $\equiv \odot$



$$\begin{cases} \mathbf{R}\cdot\mathbf{R} > 0 \\ \star\mathbf{R}\cdot\mathbf{R} = 0 \ \text{ in the orbital plane} \\ \star\mathbf{R}\cdot\mathbf{R} \neq 0 \ \text{ elsewhere} \end{cases}$$

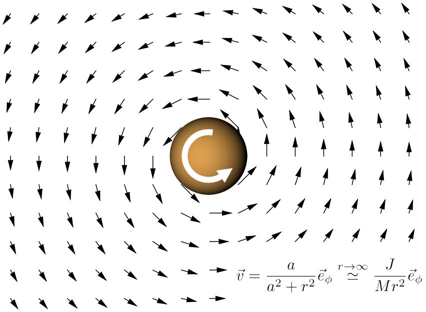
Post-Newtonian (1PN) metric, equatorial plane

- Off the Earth-Sun axis, at any point there is an unique observer for which $\mathbb{H}'_{\alpha\beta} = 0$ (similar to Petrov type I)
- On the axis, is similar to Petrov type D, and electromagnetism:
  - $\vec{v}$ has a component parallel to the super-Poynting vector $\vec{\mathcal{P}}$:

$$v \overset{r \gg r_{\odot\oplus}}{\simeq} \frac{3J}{M_\odot r}$$

    which asymptotically differs by a factor of 3/2 from the electromagnetic analogue
  - and has an *arbitrary* component along the axis

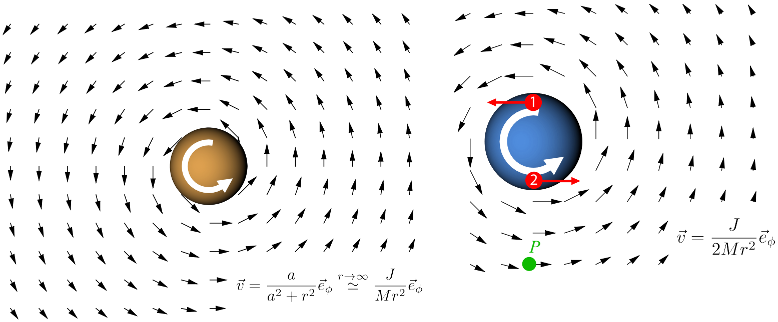
Kerr, not too close to the horizon



$$\begin{cases} \mathbf{R}\cdot\mathbf{R} > 0 \\ \star\mathbf{R}\cdot\mathbf{R} = 0 \quad \text{in the equatorial plane} \\ \star\mathbf{R}\cdot\mathbf{R} \neq 0 \quad \text{elsewhere} \end{cases}$$

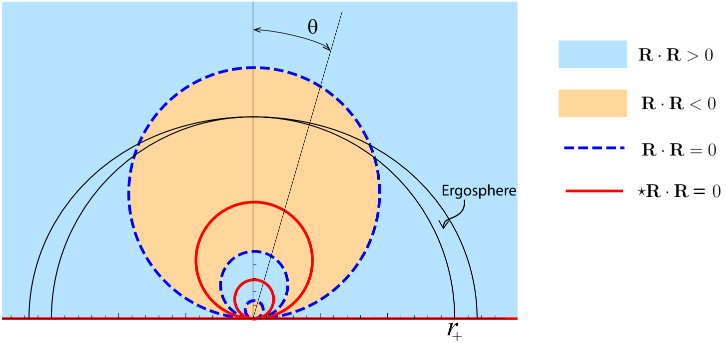
- In the equatorial plane there are observers for which $\mathbb{H}'_{\alpha\beta} = 0$, elsewhere $\mathbb{H}'_{\alpha\beta} = 0$ for all observers
- Similar to the orbital plane two body system
  - contradicts the (invariant based) claim that gravitomagnetic effects arising in the Earth-Sun system are of different nature

Kerr, not too close to the horizon



- In the equatorial plane there are observers for which $\mathbb{H}'_{\alpha\beta} = 0$, elsewhere $\mathbb{H}'_{\alpha\beta} = 0$ for all observers
- Similarly to the orbital plane two body system
  - contradicts the (invariant based) claim that gravitomagnetic effects arising in the Earth-Sun system are of different nature
- For $r \to \infty$, the velocity of the observers measuring $\mathbb{H}'_{\alpha\beta} = 0$ matches the electromagnetic counterpart up to a factor of 2

Kerr black hole



Half-plane $\theta \leq \pi/2$

- The electromagnetic analogy holds until very close to the horizon
- Inside the black holes there are both regions of electric ($\mathbf{R}\cdot\mathbf{R} > 0$) and magnetic ($\mathbf{R}\cdot\mathbf{R} < 0$) dominance
  - for sufficiently large *a*, they may lie partly inside the horizon
- There are purely electric and purely magnetic shells, where $\star\mathbf{R}\cdot\mathbf{R} = 0$ observers exist for which $\mathbb{H}'^{\alpha\beta} = 0$ or $\mathbb{E}'^{\alpha\beta} = 0$

Curvature Invariants and the gravitomagnetic effects

So far we have:

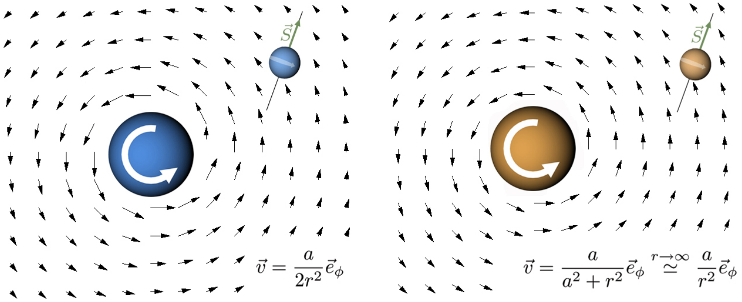
- understood the algebraic implications of the invariants
- explained the invariant structure of the systems under discussion

The remaining question:

- what are the implications for the motion of test particles?

Purely formal analogy $\{E^\alpha,\ B^\alpha\} \leftrightarrow \{\mathbb{E}^{\alpha\beta},\ \mathbb{H}^{\alpha\beta}\}$

- Gravitational tidal tensors $\{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$ are dynamically analogous to the electromagnetic tidal tensors $\{E_{\alpha\beta}, B_{\alpha\beta}\}$
- **not** to the electromagnetic **fields** $\{E^\alpha, B^\alpha\}$
- effects can be *opposite*



- No Larmor precession:

$$\vec{B} = 0 \Rightarrow \frac{D\vec{S}}{d\tau} = 0$$

- There is a Force applied:

$$F^\alpha_{\rm EM} = B^{\beta\alpha}\mu_\beta \neq 0$$

- Gyroscope precesses:

$$\frac{d\vec{S}}{d\tau} \neq 0$$

- No Force on Gyroscope:

$$\mathbb{H}_{\alpha\beta} = 0 \Rightarrow F^\alpha_{\rm G} = -\mathbb{H}^{\beta\alpha}S_\beta = 0$$

GEM inertial fields vs GEM tidal tensors

The objects that play in gravity a role analogous to the electromagnetic fields $\{E^\alpha, B^\alpha\}$ are the gravitoelectric field G^α and the gravitomagnetic field H^α

- Geodesic equation: $\frac{\tilde{D}\vec{U}}{d\tau} = \gamma\left[\gamma\vec{G} + \vec{U}\times\vec{H} - \sigma^{\hat{i}}{}_{j}U^{j}\mathbf{e}_{\hat{i}} - \frac{1}{3}\theta\vec{U}\right]$
- Gyroscope precession $\frac{d\vec{S}}{d\tau} = \frac{1}{2}\vec{S}\times\vec{H}$
- These are inertial fields:
  - $G^\alpha = -\nabla_{\mathbf{u}}u^\alpha$ is minus the observer's acceleration
  - $H^\alpha = 2\omega^\alpha$ is twice their vorticity
- H^α (not $\mathbb{H}_{\alpha\beta}$) is involved in the effects under experimental scrutiny

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- H^α (not $\mathbb{H}_{\alpha\beta}$) is involved in the effects under experimental scrutiny
- relation between the two complicated in general; at 1PN

$$\mathbb{E}_{ij} = -\nabla_j G_i + G_i G_j + \frac{1}{2}\epsilon_{ijk}\frac{\partial H^k}{\partial t} - \frac{\partial^2 U}{\partial t^2}\delta_{ij}$$

$$\mathbb{H}_{ij} = -\frac{1}{2}\nabla_j H_i - \epsilon_{ijk}\frac{\partial G^k}{\partial t}$$

- linear terms of $\{\mathbb{E}_{\alpha\beta}, \mathbb{H}_{\alpha\beta}\}$ are one order higher in differentiation compared to $\{G^\alpha, H^\alpha\}$

Pedagogical example — the Godel universe

$$ds^2 = -\left(dt + \mathcal{A}_i dx^i\right)^2 + h_{ij} dx^i dx^j \qquad \mathcal{A}_i dx^i = -e^{\sqrt{2}\omega x} dy$$
$$h_{ij} dx^i dx^j = dx^2 + \tfrac{1}{2} e^{2\sqrt{2}\omega x} dy^2 + dz^2$$

- $\mathbb{E}^{\alpha\gamma}\mathbb{H}_{\alpha\gamma} = \star\mathbf{R}\cdot\mathbf{R} = 0$ everywhere;
- with respect to the rest observers $u^\alpha = \delta^\alpha_0$ (rigid congruence):

GEM inertial fields	GEM tidal tensors
$\vec{G} = 0$	$\mathbb{E}_{\alpha\beta} \neq 0$
$\vec{H} \neq 0$	$\mathbb{H}_{\alpha\beta} = 0$

- From the point of view of the curvature it is purely electric $\Rightarrow$ gyroscopes feel no force: $F^\alpha_{\rm G} = -\mathbb{H}^{\beta\alpha} S_\beta = 0$
- From the point of GEM inertial fields, it is purely magnetic
  - gyroscopes precess $d\vec{S}/dt = \vec{S} \times \vec{H}/2$
  - It is impossible to find a rigid frame where $\vec{H} = 0$ (no rigid, vorticity-free observer congruences exist)
  - i.e., no rigid frame where Coriolis forces vanish, and gyroscopes do not precess (only at a point)

What the invariants say about GEM inertial fields

The invariants are built on the GEM tidal tensors $\mathbb{E}_{\alpha\beta}$, $\mathbb{H}_{\alpha\beta}$; it is about them that they tell us directly.
Exception is the case of Petrov type D vacua

- When $\mathbf{R} \cdot \mathbf{R} > 0$, $\star\mathbf{R} \cdot \mathbf{R}$ (purely electric condition) holds in a open 4D spacetime region, then in that region observer congruences without shear ($\sigma_{\alpha\beta} = 0$) and vorticity ($\omega^{\alpha} = 0$) exist (Wylleman-Beke 2010).
- i.e., reference frames exist relative to which frame-dragging is well defined, and $H^{\alpha} = 2\omega^{\alpha} = 0$
- This distinguishes Schwarzschild from Kerr:
  - in Schwarzschild shear-free congruences exist relative to which $H^{\alpha} = 0$, e.g. the static observers
  - in Kerr $H^{\alpha} \neq 0$ in any shear-free frame
- But is it only for Petrov type D, so does not apply to the 2-body systems
- and it is only for open 4D regions, hence does not even apply to the discussion of the purely electric equatorial plane of Kerr

Conclusion

- We clarified the algebraic meaning of the invariants as yielding conditions for the existence of observers vanishing of the electric/magnetic parts of **F** and **R**, and constructed explicitly their velocities
- we were able to understand the invariant structure of the astrophysical setups of interest, on which the electromagnetic analogy gives valuable insight
  - claims that the field invariants of a rotating and two translating sources are substantially different are unfounded
- we dissected the implications in the motion of test particles (caution with the electromagnetic analogies!)
  - The use of scalar invariants to discuss the Lense-Thirring and inertial force effects is essentially misguided.
  - Generically, curvature invariants *do not* tell us about the gravitomagnetic H^{α} itself;
  - Curvature invariants tell about the gravitomagnetic *tidal* field $\mathbb{H}_{\alpha\beta}$ (magnetic curvature)
  - Appropriate probe to measure magnetic curvature (intrinsic or extrinsic) is the *force* (*not* the precession) on a gyroscope

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- Thank you