Stability of spatially homogeneous spacetimes with a positive comological constant

Filipe C. Mena

Universidade do Minho, Portugal

Joint work with Christian Lübbe (Queen Mary, Univ. London, U.K.)

The main questions

What's is the fate of the universe?

- How far does it depend on initial conditions?
- Are the models stable? In what sense?

Observations and models

On large scale our universe is almost spatially homogeneous.

The universe undergoes an accelerated expansion, suggesting a posititve cosmological constant $\Lambda > 0$.

Questions arising

1. What is the long-term evolution of an exact homogeneous spacetime?
2. Are these predictions sensitive to perturbations, i.e. does the presence of small inhomogeneities alter the long term evolution?

The cosmic no-hair conjecture

Gibbons-Hawking, PRD, 1977

"Generic" expanding cosmological solutions to the EFEs with a positive cosmological constant Λ tend asymptotically in time to the De-Sitter solution.

Some previous results

- Spatially homogeneous solutions (Wald, PRD, 1983)
- Some inhomogeneous exact solutions with symmetries (Barrow et al, 80's)
- Linear metric perturbations (Starobinski,JETP Lett.,'79)
- Second order perturbations (Bruni, M.,Tavakol,CQG,'02)

Theorem (M., JPA, 2010)

Let $g_{\alpha\beta}^{(0)}$ be the flat RW Λ-dust metric. Then

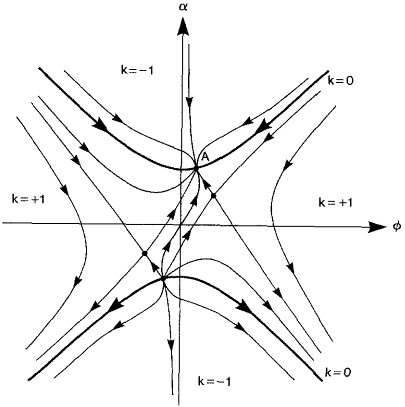
$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + \sum_{i=1}^{n} \frac{1}{i!} g_{\alpha\beta}^{(i)},$$

is convergent and approaches the De-Sitter metric locally asymptotically in time.

Asymptotic dynamics

Results with ODEs and Dynamical Systems

- Spatially homogeneous with scalar fields *(Rendall, 2004)*
- FLRW in Coley, "Dynamical Systems and Cosmology", 2003
- Linear pert. RW and Bianchi I *(Woszczyna '92; Dunsby '93, M. and Alho, '14)*



Asymptotic dynamics

Results with PDE theory

- Einstein-Maxwell-Yang-Mills fields *(Friedrich, J. Diff. Geom., 1991)*
- Non-linear stability for scalar fields *(Ringström, Invent. Math., 2008)*
- RW with $\gamma = 1/3$ *(Lübbe and Kroon, Ann. Phys., 2013)*
- RW with $0 < \gamma < 1/3$ *(Rodnianski and Speck, J.Eur.Math.Soc.,2013)*

General ideas

1. Take fully non-linear perturbation.
2. Prove
$$||g - g_{dS}|| \to 0 \quad \text{and} \quad ||K - K_{dS}|| \to 0$$
in some suitable norm.
3. Choose coordinates that bring the PDE system into a symmetric hyperbolic form for analyzing behaviour in the far future via energy estimates.

What about spatially homogeneous spacetimes?

Spatially homogeneous spacetimes

(M, g) has S_3 Lie group of isometries acting on spacelike hypersurfces Σ_t.
If group acts simply transitively, then (M, g) is called Bianchi spacetime.

Bianchi spacetimes

Lie algebra generated by the Killing vector fields ξ

$$[\xi_i, \xi_j] = C^k_{\ ij}\xi_k \,,$$
$$C^k_{\ ij} = \varepsilon_{ijl}n^{kl} + a_i\delta_j^{\ k} - a_j\delta_i^{\ k},$$

onde ε_{ijk} is the Levi-Civita symbol and n^{ij} and a^i constants.

Class	Type	n_1	n_2	n_3
A $(a = 0)$	I	0	0	0
	VIII	$-$	$+$	$+$
	IX	$+$	$+$	$+$
B $(a \neq 0)$	V	0	0	0
	IV	0	0	$+$

Table: Types I, V and IX generalize RW models $k = 0, -1$ and $+1$, resp.

Cosmic no-hair for exact Bianchi spacetimes

Theorem (Wald, 1983)

Suppose we are given a spatially homogeneous (Bianchi) spacetime

- with ${}^{(3)}R \leq 0$ (not Bianchi IX),
- that is initially expanding,
- satisfies $G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}$ with $\Lambda > 0$,
- where $T_{\mu\nu}$ satisfies the dominant and strong energy conditions

Then this Bianchi spacetime evolves exponentially towards de Sitter space.

Due to the continued expansion, we observe that at late times

- spatial geometry and matter distribution are smoothed out,
- universe homogenises and isotropises,
- universe looks locally more and more like de-Sitter space,
- no distinguishable features $\longrightarrow$ no hair.

Conformal methods in GR

Conformal compactification

$$\tilde{g} = \Theta^2 g$$

- "unphysical" or "compactified" spacetime $\tilde{M}$ has boundary B and interior M.
- Θ such that $\Theta|_B = 0$ and $d\Theta|_B \neq 0$
- $\tilde{g}$ extends as a smooth non-degenerate Lorentzian metric on $\tilde{M}$
- But conformal compactification is not always possible. Need certain decay properties

Advantages

- Global patches into local patches, i.e. get a compactified local coordinate system (can get to $\mathscr{I}^+$)
- Coordinate system non-singular (no terms $1/x$)
- Get a symmetric hyperbolic system

Stability result for RW using conformal methods

Theorem (Lübbe and Kroon, 2013)

Given Cauchy initial data for the Einstein-Euler system with $\Lambda > 0$ *and* $p = \frac{1}{3}\rho$*. If the initial data is sufficiently close to data for RW with* $p = \frac{1}{3}\mu$*, same* Λ *and spatial curvature* $k = 1$*, then*

- *the development exists globally towards the future,*
- *is future geodesically complete,*
- *remains close to the RW solution.*

- The stability result is fully non-linear.
- The result makes use of the conformal Einstein field equations (CEFE), originally due to Friedrich, adapted for radiation fluids (Einstein-Maxwell).
- The stability result can be extended as long as the reference space-time (background) is shown to be a regular solution of the CEFE.

The equations for Bianchi spacetimes with $\lambda = \sqrt{\Lambda/3}$

DEC and SEC and ${}^{(3)}R \leq 0$ imply

$$\dot{H} \leq \lambda^2 - H^2 \leq 0 \Longrightarrow \lambda \leq H \leq \lambda \cosh(\lambda t) \Longrightarrow H = \lambda + o(e^{-2\lambda t})$$

as well as

$$0 \leq 2\sigma^2 \leq 6(H^2 - \lambda^2) \leq 6\lambda^2 \sinh^{-2}(\lambda t) \Longrightarrow \sigma = o(e^{-\lambda t})$$

but this estimate is not enough.

Spacetimes non-conformally flat, so need to control the Weyl tensor

$$\begin{aligned} \partial_0(H^{\alpha\beta}) \quad = \quad & -3\theta H^{\alpha\beta} + 3\sigma^{(\alpha}{}_{\gamma} H^{\beta)\gamma} - \delta^{\alpha\beta}\sigma_{\gamma\delta} H^{\gamma\delta} \\ & +3n^{(\alpha}{}_{\gamma}(E^{\beta)\gamma} - \frac{1}{2}\pi^{\beta)\gamma}) - \frac{1}{2}n^{\gamma}{}_{\gamma}(E^{\alpha\beta} - \frac{1}{2}\pi^{\alpha\beta}) - \delta^{\alpha\beta} n_{\gamma\delta}(E^{\gamma\delta} - \frac{1}{2}\pi^{\gamma\delta}) \\ & -\varepsilon^{\gamma\delta(\alpha}(\partial_\gamma - a_\gamma)(E^{\beta)}{}_{\delta} - \frac{1}{2}\pi^{\beta)}{}_{\delta}) - \varepsilon^{\gamma\delta(\alpha}\dot{u}_\gamma E^{\beta)}_{\gamma} \end{aligned}$$

Detailed decay rates for exact Bianchi

We consider Einstein-Maxwell and Einstein-Euler systems with $\gamma = \frac{1}{3}$.

Using a $3+1$-orthonormal frame approach one get can more precise decay rates.

Here $\lambda = \sqrt{\Lambda/3}$ and $L = e^{-\lambda t}$

$$
\begin{aligned}
\lambda < H \quad &< \quad \lambda \coth(\lambda t) \\
0 < C_1 e^{\lambda t} \leq L \quad &\leq \quad C_2 e^{\lambda t} \\
n_{ab}, a_a \quad &= \quad O(L^{-1}) \\
\sigma_{ab}, {}^{(3)}R \quad &= \quad O(L^{-2}) \\
\rho, p, q_a, \pi_{ab} \quad &= \quad O(L^{-4}) \\
E_a, B_a \quad &= \quad O(L^{-2}) \\
E_{ab}, H_{ab} \quad &= \quad O(L^{-2})
\end{aligned}
$$

Regular solution to the CEFE and stability

Summary of procedure

- Rescale the metric $\tilde{g}_{\mu\nu} = \Theta^2 g_{\mu\nu}$ with $\Theta = L^{-1}$
- Define conformal time $\tau = \int_0^t \frac{1}{L(s)} \mathrm{d}s \Rightarrow \tau$ is finite at conformal infinity
- Use $\tilde{g}$-orthonormal frame $\Rightarrow$ rescaled quantities are finite at conformal infinity.
- Bianchi spacetimes here give regular solution to the CEFE.
- Use them as reference spacetimes for the stability theorems.

The stability in more detail

- Give data at $\mathscr{I}^+$ and evolve back in conformal time
- Perturb a slice Σ nonlinearly close enough to Bianchi
- Since system is symmetry hyperbolic, use Kato's theorem (ARMA,1975)

Let $||\cdot||_m$ denote a Sobolev-like norm on the space of functions on Σ. Let $m \geq 4$ and $\hat{w}_0$ the perturbation on the initial data w_0. There is ϵ such that if $||\hat{w}_0||_m < \epsilon$, then w_0 determines a unique (stable) solution w to the CEFE.

Cosmic no-hair for almost Bianchi space-times

Theorem

Given a small perturbation of a Bianchi spacetime (except type IX) whose matter content is Einstein-Maxwell or a radiation fluid, then these 'almost Bianchi' spacetimes locally asymptote to de Sitter space at late times.

- The theorem shows that Wald's result is stable.
- Almost Bianchi spacetimes (except Bianchi IX) satisfy the cosmic no-hair conjecture, i.e locally homogenise and isotropise to locally approach de Sitter at late times.
- The physical spacetime has 'lost its hair'.
- However CEFE and conformal infinity retain the information of the 'approximate Bianchi-type' and matter model (hair style) in terms of non-vanishing rescaled quantities.

Discussion and outlook

- Results are local and make no statement about the global spatial topology.
- For Bianchi IX Wald shows that if Λ is sufficiently large they also satisfy cosmic no-hair conjecture.
- If ${}^{(3)}R > 0$ or $\Lambda = 0$ then recollapse is possible (Lin & Wald 1990)
- For other trace-free energy momentum tensor (null fluids, Vlasov, conformal scalar field) the regular CEFE are expected $\rightarrow$ generalisations seem possible. But at the moment no stability theorems using the CEFE are known.
- CEFE with non-trace-free matter? Work of Oliynyk (2015) may provide the first steps..