Cosmology of the de Sitter Horndeski Models

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100 years of General Relativity

A brief cosmologist's view

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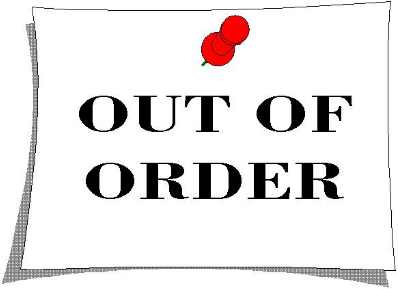

The Horndeski Lagrangian

$$\mathcal{L} = \frac{R}{2} + \sum_{i=2}^{4} \mathcal{L}_i + \mathcal{L}_m$$

It is the most general scalar field theory in 4D with second order equations of motion

- Found by Horndeski in 1975
- Rediscovered by Deffayet et al. in 2011
- Includes all other models (quintessence, k-essence, scalar-tensor, Galileon, etc.)

The Horndeski Lagrangian

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It is the most general scalar field theory in 4D with second order equations of motion

$$\begin{aligned}
\mathcal{L}_2 &= K(\phi, X) \\
\mathcal{L}_3 &= -G_3(\phi, X)\Box\phi \\
\mathcal{L}_4 &= G_4(\phi, X)R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi)\right] \\
\mathcal{L}_5 &= G_5(\phi, X)G_{\mu\nu}(\nabla^\mu\nabla^\nu\phi) - \\
&\quad \frac{1}{6}G_{5,X}\left[(\Box\phi)^3 - 3(\Box\phi)(\nabla_\mu\nabla_\nu\phi)(\nabla^\mu\nabla^\nu\phi) + \right. \\
&\quad \left. 2(\nabla^\mu\nabla_\alpha\phi)(\nabla^\alpha\nabla_\beta\phi)(\nabla^\beta\nabla_\mu\phi)\right]
\end{aligned}$$

Recipe for self-tuning Lagrangians, in concept

$$L(\phi, \dot{\phi}, a, \dot{a}) = a^3 \sum_{i=0}^{3} Z_i(\phi, \dot{\phi}, a) H^i$$

where H is Hubble rate

$$Z_i(\phi, \dot{\phi}, a) = X_i(\phi, \dot{\phi}) - \frac{k}{a^2} Y_i(\phi, \dot{\phi})$$

and X_i and Y_i are functions of the Horndeski or Deffayet free functions.

1. The theory must admit "the vacuum" for any value of the cosmological constant;
2. This should remain true before and after the phase transition where the cosmological constant jumps instantaneously by a finite amount;
3. The theory allows for a non-trivial cosmology.

Charmousis et al. 2011

Recipe for self-tuning Lagrangians, physically

We require that an abrupt change in the matter sector is absorbed by the scalar field leaving the vacuum unchanged.

1. The field equation must be trivially satisfied at the critical point to allow the field to self-adjust $(L_{\rm cp}(a, \phi, \dot{\phi}) = L_{\rm cp}(a))$;
2. At the critical point, the Hamiltonian must depend on $\dot{\phi}$ so that the continuous field can absorb discontinuities of the vacuum energy $(\mathcal{H} \propto \rho_{\rm vac} \Rightarrow \mathcal{H}_{\rm cp} \propto f(\dot{\phi}))$;
3. The scalar field equation of motion must depend on $\dot{H}$, such that the cosmological evolution is non-trivial before screening takes place $(\dot{\phi} \propto \dot{H})$.

Charmousis et al. 2011

The Fab 4

The Fab Four potentials (Charmousis *et al.*) are indeed able to self-tune for $k=-1$

$$\mathcal{L}_{\text{John}} = V_{\text{J}}(\phi)G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi,$$
$$\mathcal{L}_{\text{Paul}} = V_{\text{P}}(\phi)P^{\mu\nu\alpha\beta}\nabla_\mu\nabla_\alpha\phi\nabla_\nu\nabla_\beta\phi,$$
$$\mathcal{L}_{\text{George}} = V_{\text{G}}(\phi)R,$$
$$\mathcal{L}_{\text{Ringo}} = V_{\text{R}}(\phi)G,$$

- The cosmological models approach a patch of Minkowski with $k=-1$ when it is an attractor, and describe matter domination before that.
- $V_{\text{J}}, V_{\text{P}} \sim$"stiff fluid"; $V_{\text{G}} \sim$"radiation"; and $V_{\text{R}} \sim$"curvature".
- Unclear how to obtain a late time accelerated universe.

What are the de Sitter Horndeski models?

The most general scalar-tensor cosmological models with second order equations of motion that, regardless of the content of the Universe, have a de Sitter critical point.

Self tuning to a spatially flat de Sitter vacuum

The Lagrangian for $k = 0$,

$$L_{\rm H} = a^3 \sum_{i=0}^{3} X_i(\phi, \dot{\phi}) H^i, \qquad L_{\rm m} = -a^3 \rho_{\rm m}$$

The Hamiltonian density

$$\mathcal{H}_{\rm H} = \sum_{i=0}^{3} \left[(i-1) X_i + X_{i,\dot{\phi}} \dot{\phi} \right] H^i$$

The field equation

$$-\frac{\mathrm{d}}{\mathrm{d}t} \left[a^3 \sum_{i=0}^{3} X_{i,\dot{\phi}} H^i \right] + a^3 \sum_{i=0}^{3} X_{i,\phi} H^i = 0$$

Self tuning to a spatially flat de Sitter vacuum

At the critical point, $H_{\rm cp} = \sqrt{\Lambda}$.

The Lagrangian that at the critical point that satisfies all the constraints, i.e., $L_{\rm cp}(a, \phi, \dot{\phi}) = L_{\rm cp}(a)$ and $\mathcal{H}_{\rm cp} \propto f(\dot{\phi})$, is

$$\mathcal{L}_{\rm H}^{\rm cp} = \sum_{i=0}^{3} X_i(\phi,\, \dot{\phi})\Lambda^{i/2} = 3\sqrt{\Lambda}\, h(\phi) + \dot{\phi}\, h_{,\phi}(\phi)$$

Martin-Moruno, NJN, Lobo (2015)

Self tuning to a spatially flat de Sitter vacuum

$$\mathcal{L}_{\rm H}^{\rm cp} = \sum_{i=0}^{3} X_i(\phi,\, \dot{\phi})\Lambda^{i/2} = 3\sqrt{\Lambda}\, h(\phi) + \dot{\phi}\, h_{,\phi}(\phi)$$

What are the $X_i(\phi, \dot{\phi})$?

1. X_i are terms linear in $\dot{\phi}$

$$X_i = 3\sqrt{\Lambda} U_i(\phi) + \dot{\phi} W_i(\phi)$$

2. X_i are terms with a non-linear dependence on $\dot{\phi}$ which contribution has to vanish at the critical point, i.e., $\mathcal{L}_{\rm H}^{\rm cp} = 0$

I.
Linear models

The linear Lagrangian

Considering also matter, the linear Lagrangian and Hamiltonian are

$$L = L_{\rm EH} + L_{\rm linear} + L_{\rm m} \qquad \mathcal{H} = \mathcal{H}_{\rm EH} + \mathcal{H}_{\rm linear} + \mathcal{H}_{\rm m} = 0$$

where

$$L_{\rm linear} = a^3 \sum_i \left(3\sqrt{\Lambda}U_i(\phi) + \dot{\phi}W_i(\phi)\right) H^i$$

$i = 0, ..., 3$, subject to the constraint at the critical point,

$$\sum_i W_i(\phi)\Lambda^{i/2} = \sum_j U_{j,\phi}(\phi)\Lambda^{j/2},$$

8 functions - 1 constraint = 7 free functions $\Rightarrow$ Mag 7!

W_i and U_i are related to the the κ_j functions of the Horndeski Lagrangian and G_j functions of the Deffayet et al. functions.

Equations of motion

Together they give respectively the field equation for H' and the Friedmann equation

$$
\begin{aligned}
H' &= 3\frac{\sum_i H^i \left(\sqrt{\Lambda}\, U_{i,\phi}(\phi) - H\, W_i(\phi)\right)}{\sum_i i\, H^i W_i(\phi)} \\
\phi' &= \sqrt{\Lambda}\frac{(1-\Omega)\, H^2 - 3\sum_i (i-1)\, H^i\, U_i(\phi)}{\sum_i i\, H^{i+1} W_i(\phi)}
\end{aligned}
$$

General considerations

1. Only $W_0 \neq 0$
2. U_i, W_j pair
3. W_i, W_j pair
4. Term-by-Term model (4 potentials)
5. Tripod model (3 potentials)

1. Only $W_0 \neq 0$

$$H' = 3\frac{\sum_i H^i\left(\sqrt{\Lambda}\,U_{i,\phi}(\phi) - H\,W_i(\phi)\right)}{\sum_i i\,H^i W_i(\phi)}$$

For $W_0 \neq 0$ but $W_1 = W_2 = W_3 = 0$ then H' is ill defined. This can be understood by inspecting

$$\mathcal{H}_{\text{linear}} = \sum_i \left[3(i-1)\sqrt{\Lambda}\,U_i(\phi) + i\,\dot{\phi}\,W_i(\phi)\right] H^i$$
$$= \sum_i \left[3(i-1)\sqrt{\Lambda}\,U_i(\phi)\right] H^i$$

independent of $\dot{\phi} \Rightarrow$ The model does not screen dynamically.
Only de Sitter attractor exists.

2. W_i, U_j pair

From the constraint equation $W_i = U_{j,\phi}\Lambda^{(j-i)/2}$ and then

$$\frac{H'}{H} = -\frac{3}{i}\left[1 - \left(\frac{H}{\sqrt{\Lambda}}\right)^{j-i-1}\right]$$

which again does not depend on ϕ.

When $j - i - 1 < 0$ and $H \gg \sqrt{\Lambda}$

$$\frac{H'}{H} = -\frac{3}{i}$$

means that we recover dust for $i = 2$.

We reach de Sitter when $H \to \sqrt{\Lambda}$.

3. W_i, W_j pair

From the constraint equation $W_i = -W_{j,\phi}\Lambda^{(j-i)/2}$ and then

$$\frac{H'}{H} = -3\frac{1-(H/\sqrt{\Lambda})^{i-j}}{j-i(H/\sqrt{\Lambda})^{i-j}}$$

again independent of ϕ.

For $j > i$ and $H \gg \sqrt{\Lambda}$

$$\frac{H'}{H} = -\frac{3}{j}$$

means that we recover dust for $j = 2$.

We reach de Sitter when $H \to \sqrt{\Lambda}$.

4. Term-by-Term model

The constraint equation is satisfied for equal powers of Λ, i.e. $W_i = U_{i,\phi}$. We have then

8 functions – 4 constraints = 4 free potentials

Defining $U_{i,\phi} = \Lambda^{-i/2} V_{i,\phi}$

$$\frac{H'}{H} = -3\left(1 - \frac{\sqrt{\Lambda}}{H}\right) \frac{\sum_i (H/\sqrt{\Lambda})^i V_{i,\phi}}{\sum_i i(H/\sqrt{\Lambda})^i V_{i,\phi}}$$

Here the field (and the background matter) contributes to the dynamics of the Universe!

For $H \gg \sqrt{\Lambda}$ and only one i component dominates $\frac{H'}{H} = -\frac{3}{i}$ means that we recover dust for $i = 2$.

We reach de Sitter when $H \to \sqrt{\Lambda}$.

5. Tripod model

Let us consider the 3 potentials U_2, U_3 and W_2. The constraint equation imposes $U_{2,\phi}\Lambda + U_{3,\phi}\Lambda^{3/2} = W_2\Lambda$, then

$$\frac{H'}{H} = -\frac{3}{2}\frac{U_{2,\phi}}{W_2}\left(1 - \frac{\sqrt{\Lambda}}{H}\right)$$

For $H \gg \sqrt{\Lambda}$, $\quad \frac{H'}{H} = -\frac{3}{2}\frac{U_{2,\phi}}{W_2}$. Therefore we need:

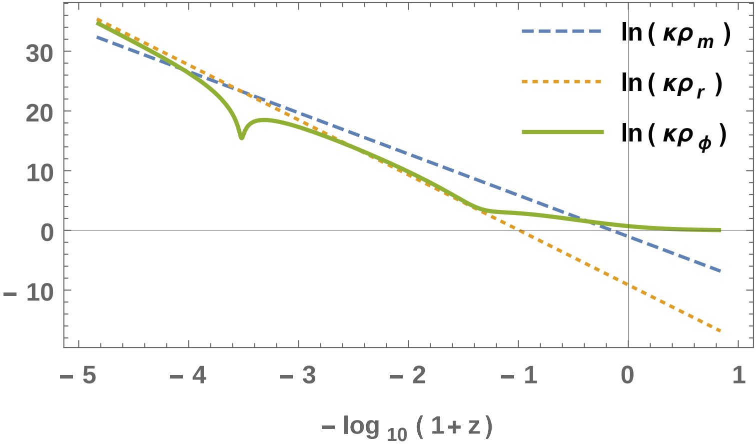
$$\frac{U_{2,\phi}}{W_2} = 1, \qquad \text{for M.D.}$$

$$\frac{U_{2,\phi}}{W_2} = \frac{4}{3}, \qquad \text{for R.D.}$$

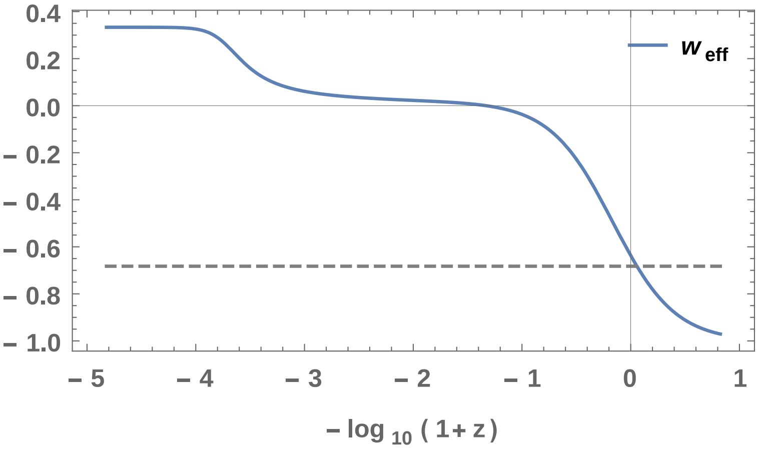
de Sitter is attained when $H \to \sqrt{\Lambda}$.

5. Tripod model: Energy densities

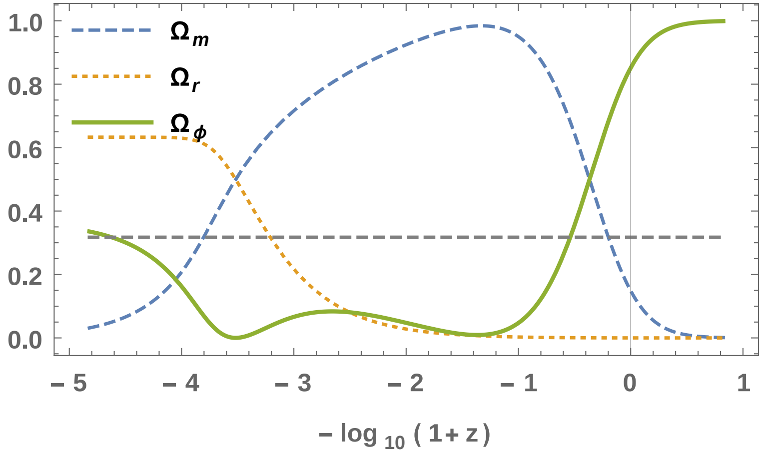
Example for: $U_2 = e^{\lambda\phi} + \frac{4}{3}e^{\beta\phi}$ and $W_2 = \lambda e^{\lambda\phi} + \beta e^{\beta\phi}$.



5. Tripod model: Effective equation of state



5. Tripod model: Abundances



Summary of Linear models

1. Only $W_0 \neq 0$
   No dynamics. Only de Sitter exists.
2. U_i, W_j pair and W_i, W_j pair
   The evolution of the Universe does not depend on the material content or on the form of the field potentials.
3. Term-by-Term model (4 potentials)
   Do depend on the field evolution but does not provide a radiation dominated epoch.
4. The tripod model (3 potentials)
   Is the most promising but the field contribution seems to be too large at early times in the studied examples.

Look for non-linear models?

II.
Non-linear models

Non-linear Lagrangian

$$L_{\rm nl} = a^3 \sum_{i=0}^{3} X_i(\phi, \dot{\phi}) H^i$$

To ensure that any non-linear dependence of the Lagrangian on $\dot{\phi}$ to vanish at the critical point,

$$\sum_{i=0}^{3} X_i(\phi, \dot{\phi}) \Lambda^{i/2} = 0$$

Again, X_i are related to the the κ_j functions of the Horndeski Lagrangian and G_j functions of the Deffayet et al. functions.

Equations of motion

Proceed to shift-symmetric case and the redefinition $\psi = \dot{\phi}$

$$H' = \frac{3(1+w)Q_0P_1 - Q_1P_0}{Q_1P_2 - Q_2P_1}$$

$$\psi' = \frac{3(1+w)Q_0P_2 - Q_2P_0}{Q_2P_1 - Q_1P_2}$$

where Q_0, Q_1, Q_2, P_0, P_1, P_2, are non-trivial functions of X_i and H, and the average equation of state parameter of matter fluids is

$$1 + w = \frac{\sum_s \Omega_s(1+w_s)}{\sum_s \Omega_s}$$

General considerations

1. $X_3 = \psi^n$ is the dominant contribution
2. $X_2 = \psi^n$ is the dominant contribution
3. X_0 and X_1 are the sole contributions
4. Extension with X_0, X_1 and X_2

1. $X_3 = \psi^n$ is the dominant contribution

When $H \gg \sqrt{\Lambda}$,

$$
\begin{aligned}
1 + w_{\text{eff}} &\simeq \frac{2}{3}(1+w), \qquad \text{for} \qquad \frac{|\,(2X_3 + \psi X_{3,\psi})\, X_{3,\psi\psi}|}{|\,(3X_{3,\psi} + \psi X_{3,\psi\psi})\, X_{3,\psi}|} \gg 1 \\
1 + w_{\text{eff}} &\simeq \frac{2}{3} \qquad \text{otherwise}
\end{aligned}
$$

Neither allow for w_{eff} corresponding to radiation and matter domination.

2. $X_2 = \psi^n$ is the dominant contribution

When $H \gg \sqrt{\Lambda}$,

$$\begin{aligned} w_{\rm eff} &\simeq w, & \text{for} \quad \frac{|\,(1 - X_2 - \psi X_{2,\psi})\, X_{2,\psi\psi}|}{|\,(2X_{2,\psi} + \psi X_{2,\psi\psi})\, X_{2,\psi}|} \gg 1, \\ w_{\rm eff} &\simeq 0, & \text{otherwise} \end{aligned}$$

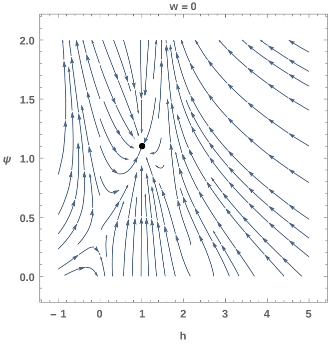
Either $w_{\rm eff}$ is too small today when compared with observational limits or, Ω_ψ is too large in the early Universe.

3. X_0 and X_1 are the sole contributions

When $H \gg \sqrt{\Lambda}$,

$$w_{\rm eff} \simeq w,$$

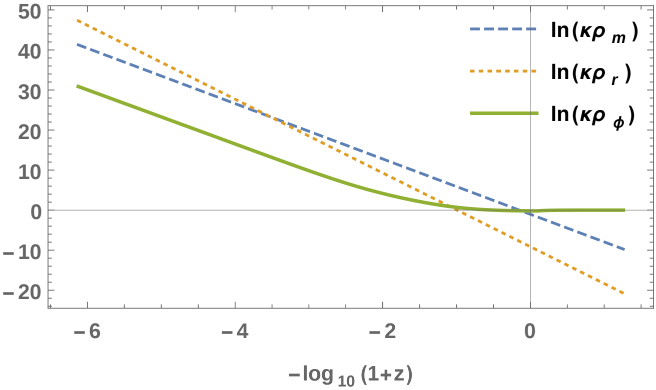
Interesting but unfortunately, models with realistic initial conditions are not driven to the critical point.



4. Extension with X_0, X_1 and X_2

Considering

$$X_2(\psi) = \alpha\psi^n, \qquad X_1(\psi) = -\alpha\psi^n + \frac{\beta}{\psi^m}, \qquad X_0(\psi) = -\frac{\beta}{\psi^m}$$

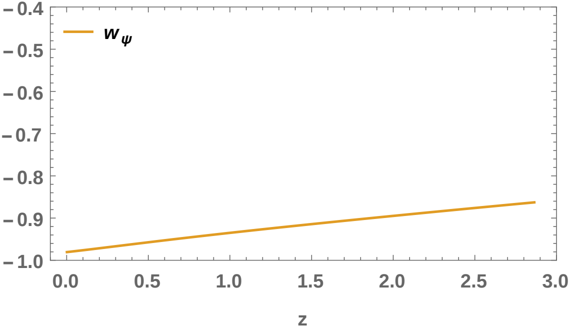


4. Extension with X_0, X_1 and X_2

Considering

$$X_2(\psi) = \alpha\psi^n, \qquad X_1(\psi) = -\alpha\psi^n + \frac{\beta}{\psi^m}, \qquad X_0(\psi) = -\frac{\beta}{\psi^m}$$

We can obtain a model with $w_\psi = w_0 + w_a(1-a)$ s.t. $w_0 = -0.98$ and $w_a = 0.04$



Summary of non-linear models

1. $X_3 = \psi^n$ is the dominant contribution
   Does not allow for $w_{\rm eff}$ corresponding to radiation and matter domination.
2. $X_2 = \psi^n$ is the dominant contribution
   $w_{\rm eff}$ is too small today when compared with observational limits or, Ω_ψ is too large in the early Universe.
3. X_0 and X_1 are the sole contributions
   Models with realistic initial conditions are not driven to the critical point.
4. Non-trivial combination of X_0, X_1 and X_2
   Can obtain a range of parameter space compatible with observational limits.

More to do

- Need study of perturbations.
- Look for combination of linear and non-linear models?