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Simulations of substructures in relativistic jets in accretion disk and black hole settings

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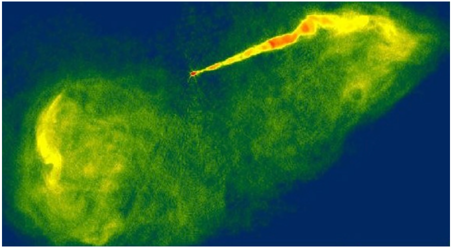
Objectives

- Formation of relativistic jets from the accretion disk into a fixed Schwarzschild BH;
- Apply unsplit finite volume methods in GRMHD.

Jet in M87

1917 Curtis: "A curious straight line connected to a core"



almost 100 years later:

jet length $\sim 230.r_s$ not fully understood

Disks and or Jets in

- Active Galactic Nuclei (AGN);
- Microquasars;
- Gamma-ray burst;
- Young stars;
- Neutron stars;
- others.

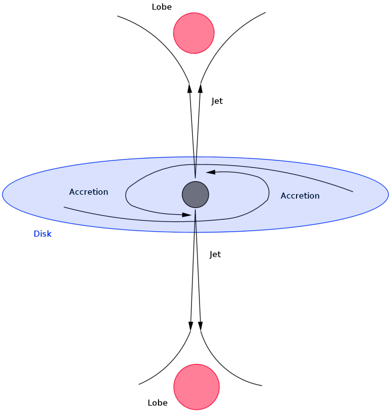
Relativistic Magnetohydrodynamics

- **1986:** Thorne, Price e MacDonald - GRMHD equations ;
- **1999:** Koide, Kudoh e Shibata - simulations - disk and jet (Lax-Wendroff);
- **2005-2006:** McNikkey , Blandford e Nishikawa - jets;
- **2010:** Koide, Kudoh e Shibata - simulations with resistive term (Lax-Wendroff);

Jet Formation

Fundamental Elements:

- Central compact object;
- Accretion rotating disk;
- Magnetosphere;
- Jet;



Basic equations

$$\nabla_\mu(\rho U^\mu) = 0$$

$$\nabla_\mu T_g^{\mu\nu} = 0$$

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$$

$$\nabla_\mu F^{\mu\nu} = -J^\nu \approx 0$$

$$T_g^{\mu\nu} = p g^{\nu\mu} + (e+p)U^\nu U^\mu + F^\mu_\sigma F^{\nu\sigma} - \frac{1}{4} g^{\mu\nu} F^{\lambda\kappa} F_{\lambda\kappa}$$

$$e = \rho + \frac{p}{\Gamma - 1}$$

Basic equations

$$ds^2 = -\alpha^2 dt^2 + \alpha^{-2} dr^2 + r^2 d\Omega^2,\ \alpha = \sqrt{1 - \frac{r_s}{r}},\ r_s = 2M$$

$$h_0 = \alpha, \quad h_1 = \alpha^{-1}, \quad h_2 = r, \quad h_3 = r \sin\theta,$$

Use tortoise coordinates in numerics

$$s = \ln\left(\frac{r}{r_s} - 1\right)$$

In this formulation, the components of the vectors **v** of velocity, **B** of magnetic field and **E** of electric field, in fiducial coordinates, are defined by

$$v_i = \frac{1}{\gamma} h_i U^i \ , \qquad (1)$$

$$B_i = \epsilon_{ijk} \frac{h_i}{J} F^{jk} \ , \qquad (2)$$

$$E_i = \frac{1}{h_0 h_i} F^{0i} \ , \qquad (3)$$

where $i, j, k \ = \ 1, 2, 3$, γ is the Lorentz factor and $J = h_1 h_2 h_3$ is the Jacobian.

The conserved quantities are given by

$$D = \gamma\rho \ , \tag{4}$$

$$\epsilon = (e+p)\gamma^2 - p - D + \frac{1}{2}\left(B^2 + E^2\right) \ , \tag{5}$$

$$\mathbf{P} = \left[(e+p)\gamma^2 \mathbf{v} + \mathbf{E}\times\mathbf{B}\right] \ , \tag{6}$$

and the stress-tension tensor,

$$T = p\mathbf{I} + (e+p)\,\gamma^2 \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B} - \mathbf{E}\mathbf{E} + \frac{1}{2}\left(B^2 + E^2\right)\mathbf{I} \ , \tag{7}$$

$$T^{ij} = h_i h_j T_g^{ij} \ , \ i,j = 1,2,3 \ .$$

Assumptions

- axial symmetry;
- variables: $x_0 = t$, $x_1 = r$ and $x_2 = \theta$;
- unknown variables: D, P_1, P_2, P_3, ϵ, B_1, B_2, B_3;

Mass Equation:

$$\frac{\partial D}{\partial t} = -\frac{1}{J}\left\{\frac{\partial}{\partial x^1}\left(h_0 h_2 h_3 D v_1\right) + \frac{\partial}{\partial x^2}\left(h_0 h_3 h_1 D v_2\right)\right\} \tag{8}$$

Energy Equation:

$$\begin{aligned}\frac{\partial \epsilon}{\partial t} = -\frac{1}{J}\left\{\frac{\partial}{\partial x^1}\left[h_0 h_2 h_3\left(P_1 - D v_1\right)\right]\right\} + \\ -\frac{1}{J}\left\{\frac{\partial}{\partial x^2}\left[h_0 h_3 h_1\left(P_2 - D v_2\right)\right]\right\} + S_5\end{aligned} \tag{9}$$

Equations of Motion:

$$\frac{\partial P_1}{\partial t} = -\frac{1}{J}\left\{\frac{\partial}{\partial x^1}\left(h_0 h_2 h_3 T^{11}\right) + \frac{\partial}{\partial x^2}\left(h_0 h_3 h_1 T^{12}\right)\right\} + S_2 \quad (10)$$

$$\frac{\partial P_2}{\partial t} = -\frac{1}{J}\left\{\frac{\partial}{\partial x^1}\left(h_0 h_2 h_3 T^{21}\right) + \frac{\partial}{\partial x^2}\left(h_0 h_3 h_1 T^{22}\right)\right\} + S_3 \quad (11)$$

$$\frac{\partial P_3}{\partial t} = -\frac{1}{J}\left\{\frac{\partial}{\partial x^1}\left(h_0 h_2 h_3 T^{31}\right) + \frac{\partial}{\partial x^2}\left(h_0 h_3 h_1 T^{32}\right)\right\} + S_4 \quad (12)$$

Magnetic Equations:

$$\frac{\partial B_1}{\partial t} = -\frac{h_1}{J}\left\{\frac{\partial}{\partial x^2}\left(h_0 h_3 E_3\right)\right\} \tag{13}$$

$$\frac{\partial B_2}{\partial t} = -\frac{h_2}{J}\left\{-\frac{\partial}{\partial x^1}\left(h_0 h_3 E_3\right)\right\} \tag{14}$$

$$\frac{\partial B_3}{\partial t} = -\frac{h_3}{J}\left\{\frac{\partial}{\partial x^1}\left(h_0 h_2 E_2\right) - \frac{\partial}{\partial x^2}\left(h_0 h_1 E_1\right)\right\} \tag{15}$$

Source Terms:

$$S_2 = h_0 \left\{ (\epsilon + D) H_{01} + H_{12} T^{21} + H_{13} T^{31} \right\} + -h_0 \left\{ H_{21} T^{22} + H_{31} T^{33} \right\} \tag{16}$$

$$S_3 = h_0 \left\{ (\epsilon + D) H_{02} + H_{23} T^{32} + H_{21} T^{12} + \right\} -h_0 \left\{ H_{32} T^{33} + H_{12} T^{11} \right\} \tag{17}$$

$$S_4 = h_0 \left\{ (\epsilon + D) H_{03} + H_{31} T^{13} + H_{32} T^{23} \right\} + -h_0 \left\{ H_{13} T^{11} + H_{23} T^{22} \right\} \tag{18}$$

$$S_5 = h_0 \left\{ H_{01} P_1 + H_{02} P_2 + H_{03} P_3 \right\} \tag{19}$$

$$H_{\mu\nu} = -\frac{1}{h_\mu h_\nu} \left(\frac{\partial}{\partial x^\nu} h_\mu \right), \ \mu, \nu = 0, 1, 2, 3$$

Electric Field (frozen-in conditions):

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}$$

Equation of state.

System of two equations (Duncan e Hughes, 1994) for x e y:

$$x(x+1)\left[\Gamma a x^2 + (2\Gamma a - b)x + \Gamma a - b + d\tfrac{\Gamma}{2}y^2\right]^2 = \\ = \left(\Gamma x^2 + 2\Gamma x + 1\right)^2\left[\tau^2(x+1)^2 + 2\sigma y + 2\sigma x y + \beta^2 y^2\right] \quad ,$$

$$\left[\Gamma(a-\beta^2)x^2 + (2\Gamma a - 2\Gamma\beta^2 - b)x + \Gamma a - b + d - \beta^2 + \tfrac{\Gamma}{2}y\right]y \\ = \sigma(x+1)(\Gamma x^2 + 2\Gamma x + 1) \quad ,$$

Newton-Raphson method, in which

$$x = \gamma - 1\ ,\ y = \gamma(\mathbf{v}\cdot\mathbf{B})\ ,\ a = D + \epsilon\ ,\ b = (\Gamma - 1)D\ ,\ d = (1 - \Gamma/2)B^2\ ,$$

$$\tau = P\ ,\ \beta = B\ ,\ \sigma = \mathbf{B}\cdot\mathbf{P}.$$

Equation of state.

Polytropic: $p = \rho^{\Gamma}$ and

$$a = \frac{p}{\rho} = \frac{\Gamma - 1}{\Gamma} \left(\frac{H}{\alpha\gamma} - 1 \right) , \tag{20}$$

$$\rho_{mag} = a^{1/(\Gamma-1)} \quad \text{and} \quad p = a^{1 + 1/(\Gamma-1)} . \tag{21}$$

Finte Volume Methods

Central

Laws of Conservation - staggered average cell: $I_i = (x_i, x_{i+1})$.

- Lax-Friedrichs;
- Nessyahu-Tadmor;
- Lax-Wendroff-Richtmyer, LWR with Runge-Kutta 3 TVD.

Godunov

Laws of Conservation - average cell: $\Omega_i = (x_{i-1/2}, x_{i+1/2})$.

- Dependency: Riemann problem on each interface;
- Riemann Solver: Harten, Lax and Leer (HLL);

Comparisons - Euler Equations:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho v) = 0 \qquad \text{(mass equation)}$$

$$\frac{\partial}{\partial t}\rho v + \frac{\partial}{\partial x}\left(\rho v^2 + p\right) = 0 \qquad \text{(motion equation)}$$

$$\frac{\partial}{\partial t}E + \frac{\partial}{\partial x}\left(v\left(E + p\right)\right) = 0 \qquad \text{(energy equation)}$$

$$\rho = \rho(x,t) \qquad \text{(density)}$$

$$v = v(x,t) \qquad \text{(velocity)}$$

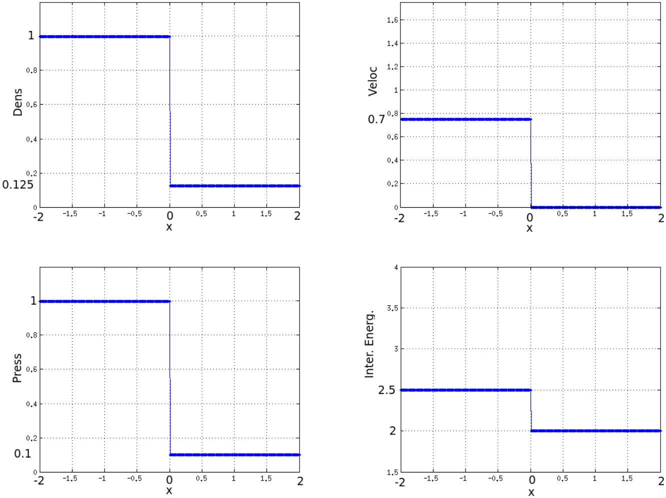
$$p = p(x,t) \qquad \text{(pressure)}$$

$$E = E(x,t) \qquad \text{(total energy)}$$

$$\epsilon = \epsilon(x,t) \qquad \text{(internal energy)}$$

with $E = \frac{1}{2}\left(\rho v^2 + \epsilon\right)$, $\epsilon = \frac{p}{(\gamma - 1)\rho}$ e γ (specific heat).

Sod Problem - Initial Conditions



Sod Problem

Table: Data

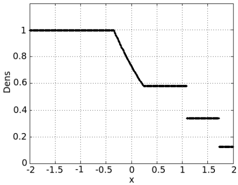
Domain	Subintervals	Δx	CFL	Δt	iterations	t_f
[-2,2]	400	0.01	0.1	0.001	800	0.8

Table: Informations of the methods

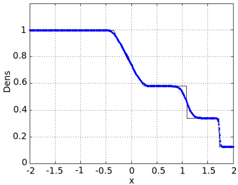
Schemes	CPU time (s)	iterations (s)
Lax-Friedrichs	31.908	0.0400
Lax-Wendroff	54.083	0.0676
Godunov-HLL	81.219	0.1015
Nessyahu-Tadmor-ST	120.790	0.1510
LWR-RK3TVD	171.841	0.2148

Simulations: Density

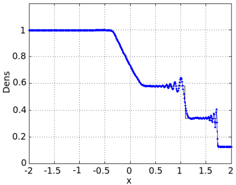
Exact



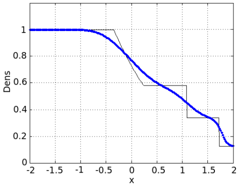
Godunov-HLL



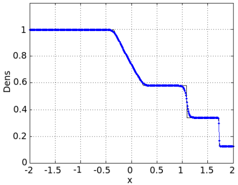
Lax-Wendroff



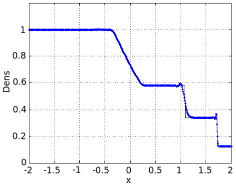
Lax-Friedrichs



Nessyahu-Tadmor



LWR-RK3-TVD



Woodward & Colella Problem - Initial Conditions

$$\rho(x,0) = \begin{cases} 1.0 & \text{if } \; x < 2 \\ 1.0 & \text{if } \; x > 2 \end{cases} \quad ,$$

$$v(x,0) = \begin{cases} 0.0 & \text{if } \; x < 2 \\ 0.0 & \text{if } \; x > 2 \end{cases}$$

and

$$p(x,0) = \begin{cases} 0.01 & \text{if } \; x < 2 \\ 1,000.00 & \text{if } \; x > 2 \end{cases} \quad .$$

Woodward & Colella Problem

Table: Data

Domain	Subintervals	Δx	t_f
[0,4]	800	0.005	0.04

Table: Informations

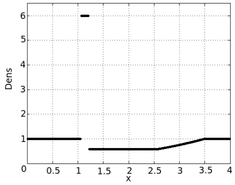
Schemes	CFL	iterations	t_f	t CPU (s)
Lax-Friedrichs	0.02	400	0.04	32.089
Godunov-HLL	0.02	400	0.04	80,471
LWR-RK3TVD (∗)	0.02	400	0.04	166.76
Nessyahu-Tadmor-ST	0.008	1,000	0.04	754.99
Lax-Wendroff (∗∗)	0.0001-0.04	-	-	-

(∗) método estável para os valores 0.01 e 0.02, e instável para 0.005 e 0.04.

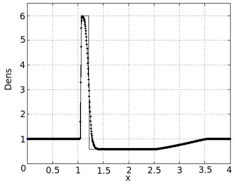
(∗∗) método não estabilizado entre 0.0001 e 0.04.

Simulations: Density

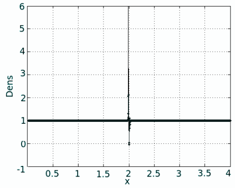
Exact



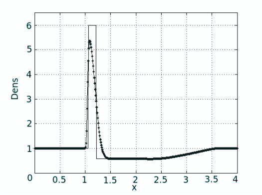
Godunov-HLL



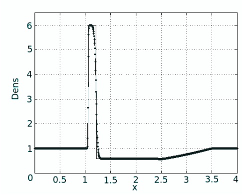
Lax-Wendroff



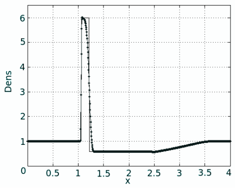
Lax-Friedrichs



Nessyahu-Tadmor



LWR-RK3-TVD



Problems with Source Term

Unidimensional problem

$$u_t + f(u)_x = s(u) \; .$$

Separating a EDP with source term in a homogeneous PDE and ODE is to define the following problems:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x,t_n) = u^n \end{cases} \Rightarrow \overline{u}^{n+1}$$

and

$$\begin{cases} \dfrac{du}{dt} = s(u) \\ u(x,t_n) = \overline{u}^n \end{cases} \Rightarrow u^{n+1} \quad .$$

Bidimensional Problems

- Numerical Methods with Split dimensional;
- Numerical Methdos without Split dimensional (Unsplit dimensional).

Nessyahu-Tadmor Method

Nessyahu-Tadmor

- Bidimensional;
- Unsplit dimensional;
- All numerical fluxes computed to the same time.

GRMHD codes

- HARM (2003);
- cosmos++ (2003);
- whyskyMHD (2007);
- ECHO (2007);
- WHARM (2007);
- CoCoA/CoCoNut (2006/2009);
- GRHydro (2013-2014).

GRMHD codes

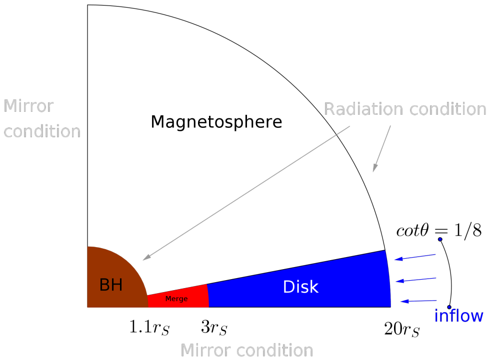
- Split methods;
- Trend: Godunov schemes - Riemann Solvers;

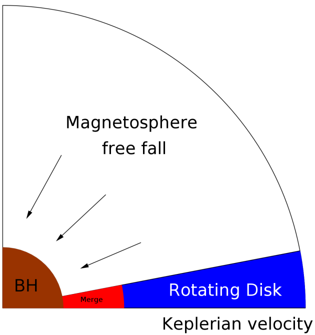
Font (2007) e Beskin (2013):

- GRMHD equations: degenerate eigenvalues;
- Recommendation: Use Unsplit dimensional methods;

Shibata et al (1999-2011): Lax-Wendroff, TVD (total variation diminishing), Split dimensional, slope limiter and artificial viscosity.

Initial and Boundary Conditions





Initial Conditions

Keplerian velocity: $v_K = \dfrac{1}{\sqrt{2\left(\dfrac{r}{r_s} - 1\right)}}$

Edge Disk: $v_K = 0.5 \Leftrightarrow r_D = 3r_s$

(relativistic disk $\rightarrow$ relativistic jet, Beskin, 2010)

Magnetic field (Wald, 1974):

$(B_r, B_\theta, B_\phi) = (B_0 \cos\theta, -\alpha B_0 \sin\theta, 0)\ ,\ B_0 = 0.3\sqrt{\rho(r_D)}$

Simulations

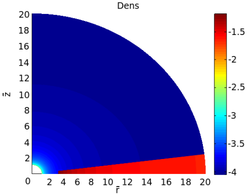
Subintervals: 630×630, $\Delta r = 18.9/210$, $\Delta\theta = \pi/420$ e $\Delta t = 0.01\Delta r$.

Constants: $\Gamma = 5/3$ e $H = 1.3$.

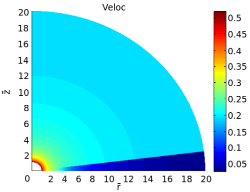
Density: $\kappa = 400$, disk 400 times denser (thicker) than magnetosphere.

Initial conditions

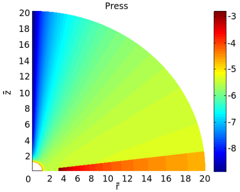
Density



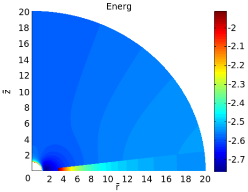
Velocity



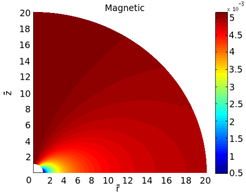
Pressure



Energy



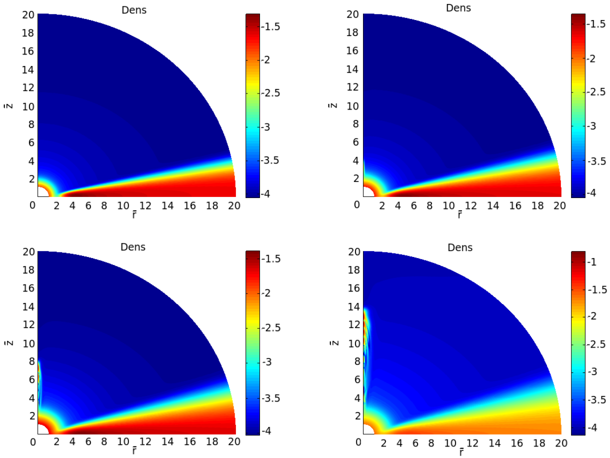
Magnetic field



Plots

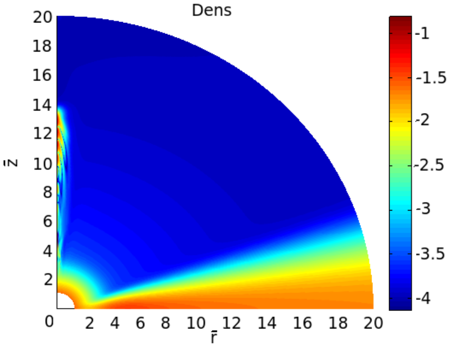
- Plot;
- Density (log10);
- Velocity;
- Pressure (log10);
- Total energy (log10);
- Magnetic field;
- iterations: 500; 1,000; 1,500 and 2,000.

Matter Density

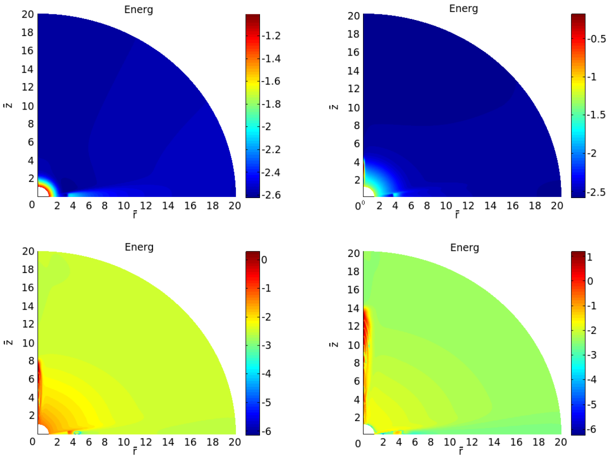


Matter Density

Video

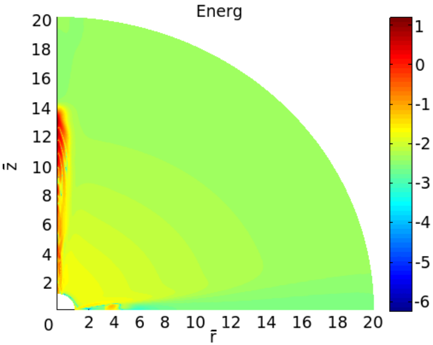


Energy Density



Energy Density

Video



Conclusions

- Central Bidimensional scheme: Nessyahu-Tadmor;
- Relativistic Jets Formation;
- Transition between disk and jet;
- Jet Length: 14 times BH radius;
- Visible substructure in the jets

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The implemented equations are the following form:

$$u_t + f(u)_x + g(u)_y = s(u) \ .$$

Separate (PDE) with no source term and then (ODE) with source term:

$$\begin{cases} u_t + f(u)_x + g(u)_y = 0 \\ u(x,y,t_n) = u^n \end{cases} \Rightarrow \overline{u}^{n+1} \tag{22}$$

and

$$\begin{cases} \frac{du}{dt} = s(u) \\ u(x,y,t_n) = \overline{u}^n \end{cases} \Rightarrow u^{n+1} \ . \tag{23}$$

Therefore, to perform a time step from t_n to t_{n+1}, we obtain an approximate solution to (22) and use this solution as the initial condition.

The equation (22) is solved by Nessyahu-Tadmor method and the solution is updated solving the equation (23) for a four-stage Euler method.

Nessyahu-Tadmor method keeps the robustness of the Lax-Friedrichs, providing stable solutions without spurious oscillations and excessive numerical dissipation.

Let $\Omega = \Omega_x \times \Omega_y$ be a regular partition of the domain spatial

$$\Omega_x : x_a = x_1 < \cdots < x_i < \cdots < x_{N+1} = x_b$$

$$\Omega_y : y_a = y_1 < \cdots < y_j < \cdots < y_{M+1} = y_b,$$

with N subintervals on x direction e M on y direction where

$$\Delta x = \frac{x_b - x_a}{N} \text{ and } \Delta y = \frac{y_b - y_a}{M},$$

The i, j-th *average cell* or *finite volume* is defined by,

$$\Omega_{i,j} = (x_{i-1/2}, x_{i+1/2}) \times (y_{j-1/2}, y_{j+1/2})$$

in which

$$x_{i+1/2} = \frac{x_{i+1} + x_i}{2}; \text{ and } y_{i+1/2} = \frac{y_{i+1} + y_i}{2}$$

Let $u = u(x, y, t)$ be a function that represents a physical quantity defined spatial domain Ω. If u is conserved in Ω, then the conservation law is satisfied on each finite volume $\Omega_{i,j}$, that is,

$$\frac{d}{dt}\int_{\Omega_{i,j}} u(x,y,t)dx = f(u(x_{i+1/2}, y_j, t)) - f(u(x_{i-1/2}, y_j, t)) + g(u(x_i, y_{j+1/2}, t)) - g(u(x_i, y_{j-1/2}, t)).$$

Regular partition independent of variable t. $\Delta t = t_{n+1} - t_n$.
Explicit algorithm for temporal evolution:

$$\begin{aligned}
&\frac{1}{\Delta x \Delta y}\int_{\Omega_{i,j}} u(x,y,t_{n+1})dx = \\
&= \frac{1}{\Delta x \Delta y}\int_{\Omega_{i,j}} u(x,y,t_n)dx + \\
&+ \frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}} f(u(x_{i-1/2},y,t))dydt + \\
&- \frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{y_{j-1/2}}^{y_{j+1/2}} f(u(x_{i+1/2},y,t)dydt + \\
&+ \frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} g(u(x,y_{j-1/2},t))dxdt +
\end{aligned} \tag{24}$$

To get a Centered Finite Volume method free of Riemann Solvers, we consider the conservation law (24) on a staggered mesh, that is,

$$I_{i,j} = (x_i, x_{x_{i+1}}) \times (y_j, y_{j+1}) \ ,$$

thus, the equation (24) is rewritten as follows

$$\frac{1}{\Delta x \Delta y}\int_{I_{i,j}} u(x,y,t_{n+1})dx = \frac{1}{\Delta x \Delta y}\int_{I_{i,j}} u(x,y,t_n)dx +$$

$$+\frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{y_j}^{y_{j+1}} \left[f(u(x_i,y,t)) - f(u(x_{i+1},y,t)\right] dydt +$$

$$+\frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}} \left[g(u(x,y_j,t)) - g(u(x,y_{j+1},t))\right] dxdt. \tag{25}$$

Consider $w = w(x, y, t)$ a piecewise bilinear polynomial function in which

$$w(x, y, t_n) = \sum_i \omega_{i,j}(x, y) p_{i,j}(x, y) \ , \tag{26}$$

with $\overline{p}_{i,j}(x_i, y_j) = \overline{w}^n_{i,j}$ and $\omega_{i,j} = 1$.

Thus, the average value of $w(x, y, t_n)$, defined on $I_{i,j}$ is given by

$$\begin{aligned}
\overline{w}^n_{i+1/2,j+1/2} = \frac{1}{\Delta x \Delta y} \int_{x_i}^{x_{i+1/2}} \int_{y_j}^{y_{j+1/2}} p_{i,j}(x, y) dy dx + \\
+ \frac{1}{\Delta x \Delta y} \int_{x_{i+1/2}}^{x_{i+1}} \int_{y_j}^{y_{j+1/2}} p_{i+1,j}(x, y) dy dx + \\
+ \frac{1}{\Delta x \Delta y} \int_{x_i}^{x_{i+1/2}} \int_{y_{j+1/2}}^{y_{j+1}} p_{i,j+1}(x, y) dy dx +
\end{aligned} \tag{27}$$

If w is an approximation to u, then from equations (27) and (25) we obtain

$$\overline{w}^{n+1}_{i+1/2,j+1/2} = \overline{w}^{n}_{i+1/2,j+1/2} +$$

$$+\frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{y_j}^{y_{j+1}} f(w(x_i,y,t))dydt+$$

$$-\frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{y_j}^{y_{j+1}} f(w(x_{i+1},y,t)dydt + \qquad (28)$$

$$+\frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}} g(w(x,y_j,t))dxdt+$$

$$-\frac{1}{\Delta x \Delta y}\int_{t_n}^{t_{n+1}}\int_{x_i}^{x_{i+1}} g(w(x,y_{j+1},t))dxdt \; .$$

As w is a piecewise bilinear polynomial, then

$$p_{i,j}(x) = \overline{w}^n_{i,j} + \frac{D_x w_{i,j}}{\Delta x}(x - x_i) + \frac{D_y w_{i,j}}{\Delta y}(y - y_j) \ , \qquad (29)$$

where $D_x w_{i,j} = MM\left\{\Delta_x w^n_{i+1/2,j} \ , \ \Delta_x w^n_{i-1/2,j}\right\}$, with
$\Delta_x w^n_{i+1/2,j} = w^n_{i+1,j} - w^n_{i,j}$ and
$D_y w_{i,j} = MM\left\{\Delta_y w^n_{i,j+1/2} \ , \ \Delta_y w^n_{i,j-1/2}\right\}$ with
$\Delta_y w^n_{i,j+1/2} = w^n_{i,j+1} - w^n_{i,j}$.
The symbol $MM\{\cdot,\cdot\}$ is the *minmod function* defined by

$$MM\{a,b\} = \frac{1}{2}\left[sign(a) + sign(b)\right]\min\{|a|,|b|\}\,, \qquad (30)$$

with $sign(a)$ being the signal of number a.

Replacing $p_{i,j}$ on equation (27), we obtain an expression for $\overline{w}^n_{i+1/2,j+1/2}$. Hence,

$$\overline{w}^n_{i+1/2,j+1/2} = \frac{1}{4}\left(\overline{w}^n_{i,j} + \overline{w}^n_{i+1,j} + \overline{w}^n_{i,j+1} + \overline{w}^n_{i+1,j+1}\right) + \\ + \frac{1}{16}\left[(D_x w_{i,j} - D_x w_{i+1,j}) + (D_x w_{i,j+1} - D_x w_{i+1,j+1})\right] + \\ + \frac{1}{16}\left[(D_y w_{i,j} - D_y w_{i,j+1}) + (D_y w_{i+1,j} - D_y w_{i+1,j+1})\right] \quad . \tag{31}$$

Use the middle point rule for the time integration and the trapezoidal rule for space integration point:

$$\int_{t_n}^{t_{n+1}} \int_{y_j}^{y_{j+1}} f(w(x_i, y, t)) dy dt \cong \cong \frac{\Delta y \Delta t}{2} \left[f\left(w_{i,j}^{n+1/2}\right) + f\left(w_{i,j+1}^{n+1/2}\right) \right] \tag{32}$$

and

$$\int_{t_n}^{t_{n+1}} \int_{x_i}^{x_{i+1}} g(w(x, y_j, t)) dx dt \cong \cong \frac{\Delta x \Delta t}{2} \left[g\left(w_{i,j}^{n+1/2}\right) + g\left(w_{i+1,j}^{n+1/2}\right) \right], \tag{33}$$

where

$$w_{i,j}^{n+1/2} = \overline{w}_{i,j}^{n} - \frac{\Delta t}{2\Delta x} D_x f_{i,j} - \frac{\Delta t}{2\Delta y} D_y g_{i,j} \ , \tag{34}$$

with

From (28) we get the **bidimensional Nessyahu-Tadmor method** on staggered mesh,

$$\overline{w}^{n+1}_{i+1/2,j+1/2} = \frac{1}{4}\left(\overline{w}^{n}_{i,j} + \overline{w}^{n}_{i+1,j} + \overline{w}^{n}_{i,j+1} + \overline{w}^{n}_{i+1,j+1}\right) +$$

$$+\frac{1}{16}\left[(D_x w_{i,j} - D_x w_{i+1,j}) + (D_x w_{i,j+1} - D_x w_{i+1,j+1}\right] +$$

$$+\frac{1}{16}\left[(D_y w_{i,j} - D_y w_{i,j+1}) + (D_y w_{i+1,j} - D_y w_{i+1,j+1})\right] +$$

$$-\frac{\Delta t}{2\Delta x}\left[f\left(w^{n+1/2}_{i+1,j}\right) - f\left(w^{n+1/2}_{i,j}\right) + f\left(w^{n+1/2}_{i+1,j+1}\right) - f\left(w^{n+1/2}_{i,j+1}\right)\right] +$$

$$-\frac{\Delta t}{2\Delta y}\left[g\left(w^{n+1/2}_{i,j+1}\right) - g\left(w^{n+1/2}_{i,j}\right) + g\left(w^{n+1/2}_{i+1,j+1}\right) - g\left(w^{n+1/2}_{i+1,j}\right)\right] . \quad (35)$$