Metric-affine theories of gravity

and the gravity-matter coupling

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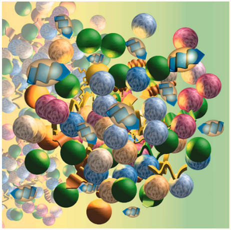
100 yy, 24 dd and some hours later...

Introduction

Introduction

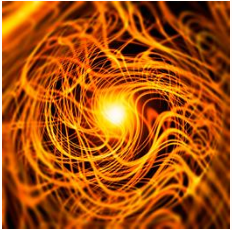
Modern theoretical physics

Strong & Electroweak



Vs.

Gravity



Extend the concepts of GR to the microphysical realm.

Introduction

"...the essential achievement of GR, namely to overcome ‘rigid’ space (ie the inertial frame), is only indirectly connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the ‘displacement field’ Γ^k_{ij}, which express the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrary separated vectors fixed by the inertial frame (ie the equality of corresponding components) by an infinitesimal operation. This makes possible to construct tensors by differentiation and hence to dispense with the introduction of ‘rigid’ space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular Γ field can be deduced from a Riemannian metric..."

A. Einstein, 4th April 1955

Introduction

In GR, spacetime geometry is fully described by the metric

Hoever, metric and the connection can be independent quantities.

Metric variation $\Rightarrow$ Palatini approach

Covariant derivative as gauge derivative of a (translation) group:

Covariant derivative of a vector:	$\nabla_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\rho} A^\rho$
Electromagnetism:	$D_\mu = \partial_\mu + ie\phi_\mu$
QCD:	$D_\mu = \partial_\mu + \frac{i}{2} g \lambda_\alpha \phi^\alpha_\mu$

Notation

The covariant derivative with respect to the independent connection $\Gamma^\gamma{}_{\alpha\beta}$ of a generic $(1,1)$-tensor is defined as

$$\overset{\Gamma}{\nabla}_\alpha A^\beta{}_\gamma = \partial_\alpha A^\beta{}_\gamma + \Gamma^\beta{}_{\alpha\delta} A^\delta{}_\gamma - \Gamma^\delta{}_{\alpha\gamma} A^\beta{}_\delta$$

Using a combination of covariant derivatives of the metric, it is quite easy to show that the connection can be decomposed as

$$\Gamma^\gamma{}_{\alpha\beta} = \left\{ {\gamma \atop \alpha\beta} \right\} + \frac{1}{2}\left(-Q_\alpha{}^\gamma{}_\beta + Q^\gamma{}_{\beta\alpha} - Q_{\beta\alpha}{}^\gamma\right) + S_{\alpha\beta}{}^\gamma - S_\beta{}^\gamma{}_\alpha + S^\gamma{}_{\alpha\beta}$$

The curvature tensor associated with the connection $\Gamma^\gamma{}_{\alpha\beta}$ is defined by

$$\mathcal{R}_{\mu\nu\rho}{}^\sigma = \partial_\nu \Gamma^\sigma{}_{\mu\rho} - \partial_\mu \Gamma^\sigma{}_{\nu\rho} + \Gamma^\alpha{}_{\mu\rho}\Gamma^\sigma{}_{\nu\alpha} - \Gamma^\alpha{}_{\nu\rho}\Gamma^\sigma{}_{\mu\alpha}$$

Some inconsistencies arising from the definition of the Ricci tensor, the scalar curvature is unambigously defined.

Bites of history....

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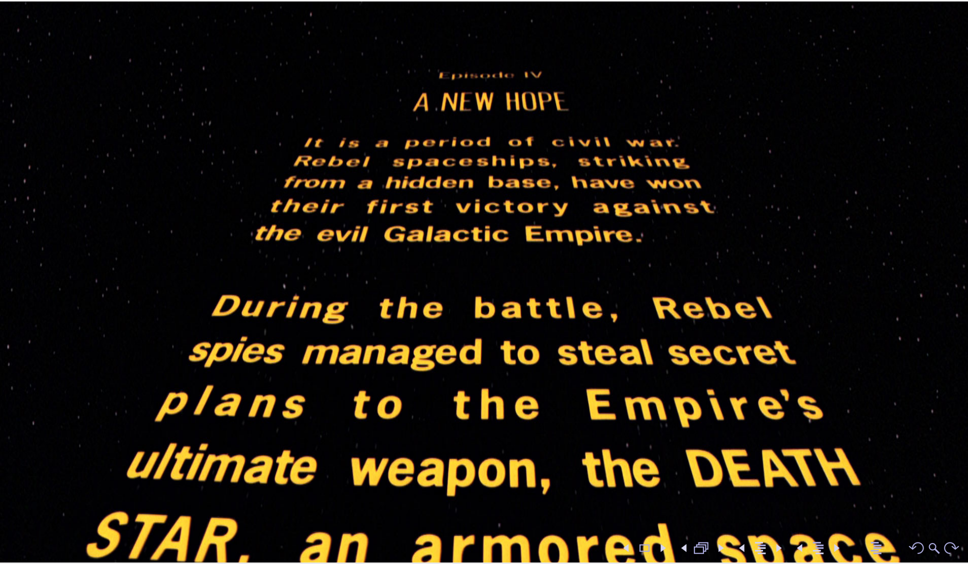

Bites of history....

Timeline of non-Riemannian generalization of GR:

1910s	Pre-non-Riemannian era	Einstein
1920s	Torsion first introduced	Einstein & Cartan
1930s	Classical Unified Theories	Einstein
1940s	Unified gravity and EM	Schroedinger
1950s	First modern gauge theory of gravity	Utiyama
1960s	Spin and local Poincaré theory	Sciama & Kibble
1970s	Local gauge theory of gravity with torsion	Hehl et al.
1980s	String theory at play	Schwarz & Scherk
1990s	Non-Riemannian structure in MAGs	Hehl, Hammond et al.
2000s	Dark Energy, Dark Matter and all that	many, many people

Getting started: Einstein-Cartan

More general framework of modified gravities: metric-affine theories.
Simplest MAG: the lesson of the ECSK theory.
$(L_4, g) \xrightarrow{Q=0} U_4 \xrightarrow{S=0} V_4 \xrightarrow{R=0} R_4$

Preferred curves in Riemann-Cartan U_4:
autoparallel vs extremals curves.

Field equations:
Einstein tensor$=k*$energy momentum
torsion$=k*$spin angular momentum

Invariance of a special relativistic theory of matter under Poincaré transformation inexorably leads to U_4!

Main consequences of U_4 theory

No waves of torsion outside
the spinning matter distribution!
But gravitational waves produced
by processes involving spin...

U_4 theory predicts a new, very weak, universal spin contact interaction of gravitational origin.

Critical mass density typically huge:

$$\rho = mn, \quad s = \hbar n/2 \quad \rightarrow \overline{\rho} = \frac{m^2}{k\hbar^2}$$

To be considered in high density regimes...

Particle pair creation when the mass density reaches
the critical density $\overline{\rho}$.

Coupling the connection to matter fields

Scalar field: no spin $\Rightarrow$ no straightforward coupling

Maxwell and non-Abelian gauge fields:
minimally coupling to torsion $\Rightarrow$ gauge symmetry breaking.

Proca field: problem of gauge non-invariance bypassed.

Coupling the connection to matter fields

Dirac field:

$$\begin{aligned} \mathcal{L}_D[\Gamma] &= (\hbar c/2)[(\overline{\nabla_\alpha \psi})\gamma^\alpha \psi - \bar{\psi}\gamma^\alpha \nabla_\alpha \psi - 2(mc/\hbar)\bar{\psi}\psi] \\ &= \mathcal{L}_D[\{\}] + \mathrm{Spin}\otimes\mathrm{Torsion} \end{aligned}$$

$$\gamma^\alpha \nabla_\alpha \psi + \frac{3}{8} l_P^2 (\bar{\psi}\gamma_5\gamma^\alpha\psi)\gamma_5\gamma_\alpha\psi + (mc/\hbar)\psi = 0$$

Neutrinos: Dirac with no spin contact term.

Standard Model Extension: low-energy limit of a physically relevant fundamental theory with Lorentz-covariant dynamics in which spontaneous Lorentz violation occurs [anyone of the 23 Kostelecky's PRLs]

Poincaré gauge theories

Utiyama attempt (1955): gravity is the gauge theory of the homogeneous Lorentz group.
Local Lorentz transformation: $x^k \to x^k + \omega^k_m(x)\, x^m$
Tetrad transforms accordingly: $e^\mu_m \to e^\mu_m - \omega^a_m(x)\, e^\mu_a$
invariance is resurrected when $\psi_{,\mu} \to \psi_{,\mu} - \frac{1}{2} A^{ab}{}_\mu \omega_{ab} \psi$

Major improvement: take the underlying symmetry group to be the inhomogeneous Lorentz group $x^\kappa \to x^\kappa + \omega^\kappa_\mu\, x^\mu + \epsilon^\kappa$

Active global Poincaré transformation

$$\phi \to (1 - \epsilon^\mu \partial_\mu + \omega^{\alpha\beta} f_{\beta\alpha})\psi$$

"localize" the theory assuming $\epsilon^\mu(x)$ and $\omega^{\alpha\beta}(x)$...

$$\partial_\mu \to D_\mu - \Gamma^{ab}_\mu f_{ab}$$

The general action

The general action will be of the form

$$\mathcal{S} = \int d^4x \sqrt{-g}[\mathcal{L}_G(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta}) + \mathcal{L}_M(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta}, \psi)]$$

Looking for the lowest order theory associated with $\mathcal{L}_G(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta})$:
power counting of gravitational terms
connection$\sim [\mathrm{length}]^{-1}$, Ricci$\sim [\mathrm{length}]^{-2}$, $\mathcal{L}_G \sim [\mathrm{length}]^{-4}$.

The most general gravitational Lagrangian density of the kind $(16\pi L_P^2)^{-1} * [\mathrm{length}]^{-2}$ has the form

$$\mathcal{L}_G = \frac{1}{16\pi L_P^2}\left[\mathcal{R} + \sum_i a_i Q_{(i)}^2 + \sum_i b_i Q_{(i)} * S_{(i)} + \sum_i c_i S_{(i)}^2\right]$$

The general action

- 5 Pure nonmetricity terms $Q_{\lambda\mu\nu} * Q_{\gamma\alpha\beta} * g^{\circ\circ} * g^{\circ\circ} * g^{\circ\circ}$
- 3 Mixed terms $Q_{\lambda\mu\nu} * S_{\alpha\beta}{}^{\gamma} * \delta^{\circ}{}_{\circ} * g^{\circ\circ} * g^{\circ\circ}$
- 3 Pure torsion terms $S_{\mu\nu}{}^{\rho} * S_{\alpha\beta}{}^{\gamma} * \delta^{\circ}{}_{\circ} * \delta^{\circ}{}_{\circ} * g^{\circ\circ}$

No covariant derivative of nonmetricity (or torsion):

$$\overset{\Gamma}{\nabla}_{\mu} Q_{\alpha\beta\gamma} = \overset{\{\}}{\nabla}_{\mu} Q_{\alpha\beta\gamma} + \text{contractions } K * Q$$

No "metric" Ricci scalar:

$$\int d^4x\sqrt{-g}R = \int d^4x\sqrt{-g}[\mathcal{R} - (K^{\alpha}{}_{\rho\alpha}K^{\rho}{}_{\mu}{}^{\mu} + K^{\alpha}{}_{\rho\mu}K^{\rho\mu}{}_{\alpha})]$$

No parity violating terms obtained contracting with $\epsilon^{\alpha\beta\gamma\delta}$

The general action

Torsion and nonmetricity are shown to be both algebraically expressed in terms of hypermomentum tensor describing the intrinsic properties of matter as spin angular momentum, shear and dilation current $\Delta^{\mu\nu}{}_{\lambda} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_M(g,\Gamma,\psi)}{\delta \Gamma^{\lambda}{}_{\mu\nu}}$

$$S_{\mu\nu\lambda} = [\hat{f}(\Delta)]_{[\mu\nu]\lambda} \quad Q_{\mu\nu\lambda} = [\bar{f}(\Delta)]_{\mu(\nu\lambda)}$$

[Vitagliano, CQG14]

The total connection is reduced to a non trivial expression of metric with its derivatives and of matter fields Δs

Consequences: the effective stress energy tensor of the (metric) field equation carries extra hypermomentum contribution (simplified cases:
Einstein-Cartan [Hehl et al, RevModPhys76]
2nd order MAG with only torsion [Vitagliano et al, AoP11])

Higher orders

Next-order invariants (with dimension $[\mathrm{length}]^{-4}$) introduce further dof, even if matter does not couple to the connection.
Ex. $\mathcal{R} + \mathcal{R}_{\mu\nu}\mathcal{R}_{\kappa\lambda}(ag^{\mu\kappa}g^{\nu\lambda} + bg^{\mu\lambda}g^{\nu\kappa})$, equivalent to GR plus a (dynamical) Proca vector field. [Vitagliano et al, PRD10]

Generalized $f(\mathcal{R})$ theories

Problem of invariance under projective transformation

$$\Gamma^{\rho}{}_{\mu\nu} \to \Gamma^{\rho}{}_{\mu\nu} + \delta^{\rho}{}_{\mu}\xi_{\nu}$$

The grav sector must break the invariance because the matter sector does! 4 dof to fix constraining the connection⇒
Lagrange multiplier build from either nonmetricity or torsion

$$A^{\mu}Q_{\mu} \equiv A^{\mu}g^{\alpha\beta}Q_{\mu\alpha\beta},\ B^{\mu}\widetilde{Q}_{\mu} \equiv B^{\mu}g^{\alpha\beta}Q_{\alpha\beta\mu},\ C^{\mu}S_{\mu} \equiv C^{\mu}S_{\mu\rho}{}^{\rho}$$

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Higher orders

$$\mathcal{S} = \int d^4x\sqrt{-g}(f(\mathcal{R}) + C^\mu S_\mu) + \mathcal{S}_M(g_{\mu\nu}, \Gamma^\gamma{}_{\alpha\beta}, \psi)$$

whose field equations are written as

$$f'(\mathcal{R})\mathcal{R}_{(\mu\nu)} - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = (8\pi L_P^2)T_{\mu\nu}$$

$$-\overset{\Gamma}{\nabla}_\lambda \left(\sqrt{-g}f'(\mathcal{R})g^{\mu\nu}\right) + \overset{\Gamma}{\nabla}_\sigma \left(\sqrt{-g}f'(\mathcal{R})g^{\sigma\mu}\right)\delta^\nu{}_\lambda +$$

$$+2\sqrt{-g}f'(\mathcal{R})g^{\mu\sigma}S_{\sigma\lambda}{}^\nu = \sqrt{-g}\left(\Delta^{\mu\nu}{}_\lambda - (2/3)\Delta^{\sigma[\nu}{}_\sigma\delta^{\mu]}{}_\lambda\right)$$

$$S_{\alpha\mu}{}^\alpha = 0$$

Fully consistent choice: for vacuum solutions the field equations in Palatini-$f(\mathcal{R})$ are recovered

Bonus feature: no propagation of torsion waves in vacuum.

$$S_{\mu\nu}{}^\lambda = \frac{1}{f'(\mathcal{R})}g^{\rho\lambda}(\Delta_{[\rho\mu]\nu} + \Delta_{[\nu\rho]\mu} - \Delta_{[\mu\nu]\rho})$$

Equations of motion

The equation of motion of a particle in general relativity (in its original 1915 form) can be obtained in at least three ways.

I.*postulate* that the particle follows a geodesics

$$\delta m \int ds = 0 \Longrightarrow \frac{dv^\sigma}{d\tau} + v^\alpha v^\beta \{^{\ \sigma}_{\alpha\beta}\} = 0$$

II.*assume* that the motion has the velocity parallel transported along the curves

$$\frac{dv^\sigma}{d\tau} + v^\alpha v^\beta \Gamma^{\ \sigma}_{\alpha\beta} = 0$$

!!in general not correct!!

Equations of motion

III. from field equations together with Bianchi (or Noether) identities

$$\frac{d}{d\tau}\frac{M^{\mu 0}}{v^0} + M^{\alpha\beta}\{^{\sigma}_{\alpha\beta}\} = M^{\eta\alpha\beta}\{^{\sigma}_{\alpha\beta}\}_{,\eta}$$

A new guest: the rotational equation of motion.
Many theories predict in the weak field limit

$$\frac{dS}{dt} \sim S\,x\,b$$

These effects may lead to experimental observations of the non-Riemannian part.

Effects of MAGs

Particle physics and laboratory tests

Large scale tests: PPN formalism of Poincaré theories. The total antisymmetric part of torsion enters as $\mathcal{O}(\epsilon^3)$ contribution.

Astrophysical sources:

Spinning neutron polarized stars: the magnetic field may arise from, or be partially due to, a ferromagnetic state in which the neutrons are aligned. This alignment would result in the creation of a non-Riemannian field. Decreasing of the angle between rotational axis and the polar magnetic axis of a pulsar.

Summary

MAGs make straightforward the coupling between geometry and internal degrees of freedom of matter fields (intrinsic spin, dilation current and shear)

We showed that the most general second order theory, the connection can be algebraically eliminated, i.e. torsion and nonmetricity only gain dynamics at higher order

Re-analysing the peculiar case of $f(\mathcal{R})$ in light of the above, problems due to projective invariance arise
$\Rightarrow$ MAG generalization of $f(\mathcal{R})$: excitation of extra dof residing in the connection? matter interaction below dynamical regime?

Enjoy Lisbon and its surprises...

(!!!but only AFTER the meeting!!!)