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Non-minimal Einstein-Maxwell theory: Petrov classification of a trace-free susceptibility tensor and Fresnel equations

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Outline

- Basic formalism of the medium electrodynamics
- Fresnel equation and its geometric interpretation
- Non-minimal Einstein-Maxwell model with traceless susceptibility tensor
- Newman-Penrose formalism
- Properties associated with Fresnel equations
- Canonical form of dispersion relations, indicatrices, and optical metrics.

Basic formalism of the medium electrodynamics

The Einstein-Maxwell theory deals with the action functional

$$S_{(\mathrm{EM})} = \int d^4x \sqrt{-g} \ \left(R + \frac{1}{2} C^{ikmn} F_{ik} F_{mn} \right) \,, \tag{1}$$

The information concerning specific features of interactions in the electromagnetically active medium (or quasi-medium) is encoded in the linear response tensor C^{ikmn}. Due to the structure of the second term in (1) this tensor possesses evident symmetry of indices

$$C^{ikmn} = -C^{kimn} = C^{mnik} = -C^{iknm} \,. \tag{2}$$

Variation of the action functional (1) with respect to potential A_i yields the electrodynamic equations

$$\nabla_k \left(C^{ikmn} F_{mn} \right) = 0 \,. \tag{3}$$

In the vacuum the linear response tensor has the simplest form

$$C^{ikmn}_{(\mathrm{vac})} = g^{ikmn} \equiv \frac{1}{2} (g^{im} g^{kn} - g^{in} g^{km}) \,. \tag{4}$$

The difference $\chi^{ikmn} \equiv C^{ikmn} - g^{ikmn}$ is called the susceptibility tensor.

Fresnel equation

In the geometrical optics approach the potential four-vector and the field strength tensor can be represented, respectively, as

$$A_p = a_p e^{i\Theta}\,, \quad F_{pq} = i(k_p A_q - k_q A_p)\,, \tag{5}$$

where Θ is the phase, a_p is a slowly varying amplitude, and k_p is a wave four-vector, the gradient of the phase: $k_m = \nabla_m \Theta$.

In the leading-order approximation, the Maxwell equations can be reduced to the system of algebraic equations

$$C_{pqrs} k^q k^r a^s = 0\,. \tag{6}$$

This set of linear equations with respect to a_s admits nontrivial ($a_s \not\propto k_s$) solutions, when the four components of k_p satisfy the dispersion (or Fresnel) equation:

$$\mathcal{G}^{pqrs} k_p k_q k_r k_s = 0\,, \quad \mathcal{G}^{pqrs} = -\frac{4}{3} C^{ipmq} C^{krns}\, {}^*C^*_{ikmn}\,, \tag{7}$$

where ${}^*C^*_{ikmn} \equiv \frac{1}{4}\epsilon_{ikls} C^{lspq} \epsilon_{mnpq}$ is a double-dual linear response tensor.

The completely symmetric tensor $\mathcal{G}^{(pqrs)}$ is known as the Tamm-Rubilar tensor.

Geometric interpretation

The dispersion relation

$$T(k) \equiv \mathcal{G}^{pqrs} k_p k_q k_r k_s = 0 \tag{8}$$

is a quartic homogeneous equation in the wave vector components k_s. This equation gives the wave vector k^i up to a factor.

In a geometric point of view, the dispersion equation $T(k) = 0$ defines a surface in a three-dimensional projective space $\mathbb{R}P^3$.

Four conditions

$$\frac{\partial T(k)}{\partial k_s} = \mathcal{G}^{(pqrs)} k_p k_q k_r = 0 \tag{9}$$

determine positions of singular points of this surface.

The classification of types of the dispersion equations is firmly associated with the classification of quartic surfaces in $\mathbb{R}P^3$.

Non-minimal Einstein-Maxwell model

The action functional is of the form

$$S_{\rm (NMEM)} = \int d^4x \sqrt{-g}\ \left\{ R + \frac{1}{2} F_{ik} F^{ik} + \frac{1}{2}\, \chi^{ikmn} F_{ik} F_{mn} \right\} , \tag{10}$$

The non-minimal susceptibility tensor χ^{ikmn} is defined as follows

$$\begin{aligned} \chi^{ikmn} \equiv\ & \frac{q_1}{2} R \left(g^{im} g^{kn} - g^{in} g^{km} \right) + \\ +\ & \frac{q_2}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}) + q_3 R^{ikmn} , \end{aligned} \tag{11}$$

where q_1, q_2, q_3 are the phenomenological parameters describing the non-minimal coupling of electromagnetic and gravitational fields.

The variation of the action functional with respect to potential A_i yields

$$\nabla_i (C^{ikmn} F_{mn}) = 0 \,,$$

$$C_{ikmn} = \frac{1}{2} (g_{im} g_{kn} - g_{in} g_{km}) + \chi_{ikmn} \,.$$

We can describe the non-minimal Einstein-Maxwell model in terms of (quasi)medium electrodynamics, where C^{ikmn} corresponds to the linear response tensor.

Traceless susceptibility tensor

The main purpose of this work is to classify the dispersion relations for the case

$$\chi_{pqrs}\, g^{qs} = 0\,, \tag{12}$$

when the susceptibility tensor is traceless.

Suitable situations:

1. The Ricci tensor vanishes, $R_{pq} = 0$; for example, Schwarzschild space-time, Kerr space-time, etc.
2. The susceptibility tensor is proportional to the Weyl tensor,

$$\chi_{pqrs} = qW_{pqrs}, \qquad q_1 = \frac{q}{3},\ q_2 = -q,\ q_3 = q.$$

For these models, we can apply Petrov's scheme to classify the tensor χ_{ikmn} and the dispersion relations.

Newman-Penrose formalism

In order to realize the Petrov classification scheme, we will follow the standard Newman-Penrose formalism.

Newman-Penrose null tetrad:

$$e^p_1 = l^p \,, \quad e^p_2 = m^p \,, \quad e^p_3 = \bar{m}^p \,, \quad e^p_4 = n^p \,,$$

$$l^p l_p = n^p n_p = m^p m_p = \bar{m}^p \bar{m}_p = 0 \,, \quad l^p m_p = l^p \bar{m}_p = n^p m_p = n^p \bar{m}_p = 0 \,,$$

$$l^p n_p = 1 \,, \quad m^p \bar{m}_p = -1$$

Five scalars related to the susceptibility tensor:

$$\Psi_0 = -\chi_{pqrs} l^p m^q l^r m^s \,, \quad \Psi_1 = -\chi_{pqrs} l^p n^q l^r m^s \,, \quad \Psi_2 = -\chi_{pqrs} l^p m^q \bar{m}^r n^s \,,$$

$$\Psi_3 = -\chi_{pqrs} l^p n^q \bar{m}^r n^s \,, \quad \Psi_4 = -\chi_{pqrs} n^p \bar{m}^q n^r \bar{m}^s$$

Space-time metric:

$$g^{pq} = l^p n^q + l^q n^p - m^p \bar{m}^q - m^q \bar{m}^p$$

Properties of the Tamm-Rubilar tensor-1

The tensor $\mathcal{G}^{(pqrs)}$ constructed from the traceless susceptibility tensor has 25 independent components and can be represented as

$$\mathcal{G}^{(pqrs)} = [1 - 2\,\mathrm{Re}(I) - 4\,\mathrm{Re}(J)]\; g^{(pq}g^{rs)} + \mathcal{H}^{(pqrs)},$$

where $\mathcal{H}^{(pqrs)}g_{pq} = 0$ and I, J are the invariants of the susceptibility tensor:

$$I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3\Psi_2^2, \qquad J = \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix},$$

$$\mathrm{Re}(I) = \frac{1}{16}\,\chi_{ikmn}\chi^{ikmn}, \qquad \mathrm{Re}(J) = -\frac{1}{96}\,\chi_{ikmn}\chi^{ikpq}\chi_{pq}{}^{mn}.$$

The tensor $\mathcal{H}^{(pqrs)}$ satisfies the indentity

$$\mathcal{H}^{(pqrs)}\mathcal{H}_{(pqrm)} = \left[\frac{16}{3}|I|^2(2\,\mathrm{Re}(I) - 3) + 32|I + 3J|^2\right]\,\delta^s_m.$$

Properties of the Tamm-Rubilar tensor-2

a) If $\Psi_0 = 0$, then $T(l) = \mathcal{G}^{pqrs} l_p l_q l_r l_s = 0$;

b) If $\Psi_4 = 0$, then $T(n) = \mathcal{G}^{pqrs} n_p n_q n_r n_s = 0$;

c) If $\Psi_0 = \Psi_1 = 0$, then $\left. \dfrac{\partial T(k)}{\partial k_s} \right|_{k=l} = \mathcal{G}^{(pqrs)} l_p l_q l_r = 0$;

d) If $\Psi_3 = \Psi_4 = 0$, then $\left. \dfrac{\partial T(k)}{\partial k_s} \right|_{k=n} = \mathcal{G}^{(pqrs)} n_p n_q n_r = 0$.

When

$$\mathcal{G}^{pqrs} k_p k_q k_r k_s = (g_A^{pq} k_p k_q) \cdot (g_B^{rs} k_r k_s) \,, \tag{13}$$

tensors g_A^{pq}, g_B^{pq} are called as optical metrics. Light propagates along null geodesics of these auxiliary space-times.

Indicatrices

In order to visualize solutions to the Fresnel equation we will draw optic indicatrices. We will define

$$k_1 = \frac{1-x}{\sqrt{2}}, \quad k_2 = \frac{y}{\sqrt{2}}\,\mathrm{e}^{i\varphi}, \quad k_3 = \frac{y}{\sqrt{2}}\,\mathrm{e}^{-i\varphi}, \quad k_4 = \frac{1+x}{\sqrt{2}}.$$

Here $k_1 = k_p l^p$, $k_2 = k_p m^p$, $k_3 = k_p \bar{m}^p$, $k_4 = k_p n^p$ are the tetrad components of the wave four-vector.

Singular points of the surface $T(k) = 0$ relates to singular points (or intersection points) of the corresponding indicatrix.

As a result:

Type O — no intersection points,
Type N — one intersection point,
Types D and III — two points,
Type II — three points,
Type I — four points.

When the optical metrics exist, the indicatrix (quartic surface) splits into two quadric surfaces. Examples can be found below.

Type O

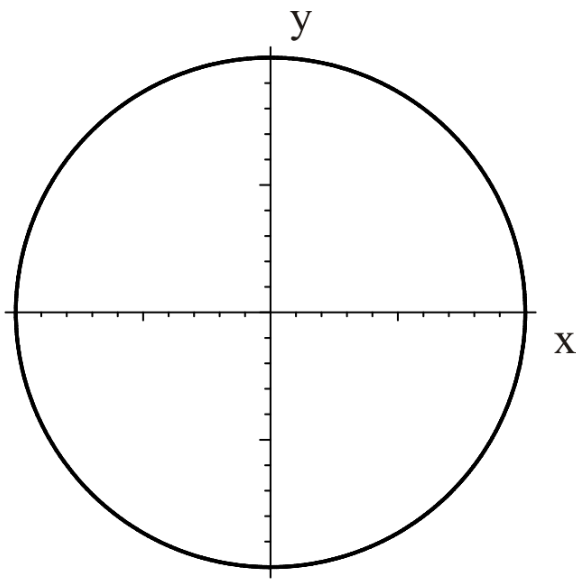
$(\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0)$

When the susceptibility tensor χ_{pqrs} vanishes, the tensor $\mathcal{G}^{(pqrs)}$ and the dispersion equation take the simplest form

$$\mathcal{G}^{(pqrs)} = g^{(pq}g^{rs)}\,,$$

$$\mathcal{G}^{(pqrs)}k_p k_q k_r k_s = (2k_1k_4 - 2k_2k_3)^2 = 0\,.$$

Plot of the indicatrix (φ is arbitrary):



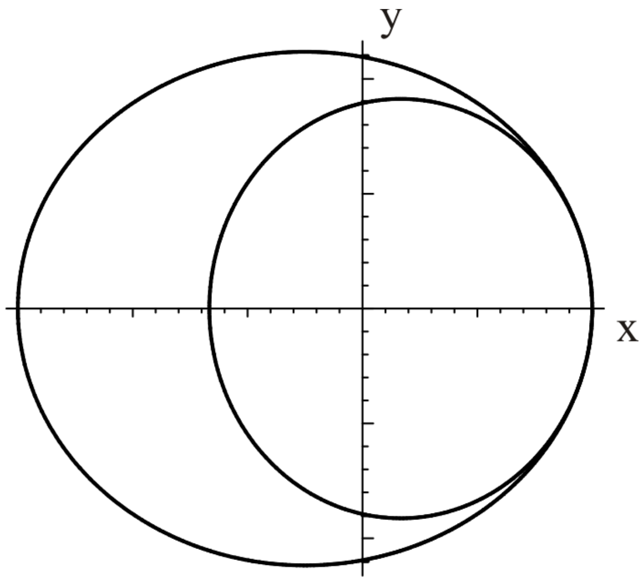
Type N

$$(\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,, \quad \Psi_4 \neq 0)$$

Canonical form of the Fresnel equation:

$$\mathcal{G}^{(pqrs)} k_p k_q k_r k_s = (2k_1 k_4 - 2k_2 k_3 + 2|\Psi_4| k_1^2)(2k_1 k_4 - 2k_2 k_3 - 2|\Psi_4| k_1^2) = 0.$$

Plot of a typical indicatrix (φ is arbitrary):



For the type N, the quartic surface splits into two ellipsoids of revolution. We deal with the birefringence phenomenon in this case.

Type N

$$(\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0\,, \quad \Psi_4 \neq 0)$$

Optical metrics:

$$g_A^{pq} = g^{pq} + 2\,|\Psi_4|\, l^p\, l^q = l^p n^q + l^q n^p - m^p \bar{m}^q - m^q \bar{m}^p + 2\,|\Psi_4|\, l^p\, l^q\,,$$

$$g_B^{pq} = g^{pq} - 2\,|\Psi_4|\, l^p\, l^q = l^p n^q + l^q n^p - m^p \bar{m}^q - m^q \bar{m}^p - 2\,|\Psi_4|\, l^p\, l^q\,.$$

l^p — the principal null direction of the susceptibility tensor

$$\chi_{pqrs} l^s = 0\,.$$

Polarization vectors ($k_i \neq l_i$):

$$A:\ a_i = k_p \left[\Psi_4 (m^p l_i - l^p m_i) + |\Psi_4| (\bar{m}^p l_i - l^p \bar{m}_i)\right],$$

$$B:\ a_i = k_p \left[-\Psi_4 (m^p l_i - l^p m_i) + |\Psi_4| (\bar{m}^p l_i - l^p \bar{m}_i)\right].$$

These vectors are non-null and orthogonal to each other:

$$g_{ik} a_A^i a_A^k \neq 0, \quad g_{ik} a_B^i a_B^k \neq 0, \quad g_{ik} a_A^i a_B^k = 0.$$

Type D

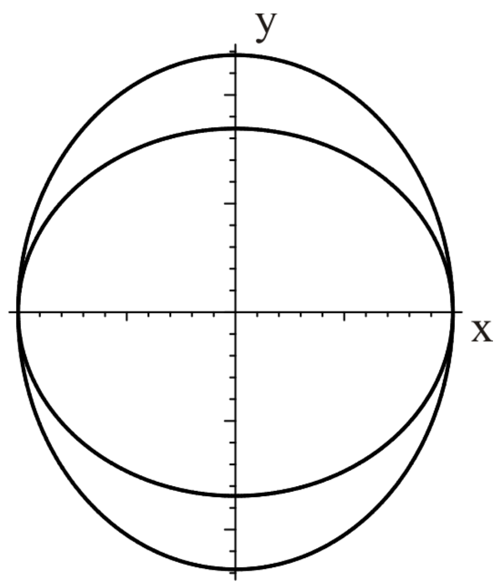
$$(\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0\,, \quad \Psi_2 \neq 0)$$

Canonical form of the Fresnel equation:

$$\mathcal{G}^{(pqrs)} k_p k_q k_r k_s = (1 - 2\,\mathrm{Re}\,\Psi_2)^2\,(1 + 4\,\mathrm{Re}\,\Psi_2)\,(2k_1k_4 - 2\mu k_2 k_3)(2k_1k_4 - \frac{2}{\mu}k_2k_3) = 0,$$

$$\mu = \frac{|1 + 2\Psi_2 - \bar{\Psi}_2| + 3|\Psi_2|}{|1 + 2\Psi_2 - \bar{\Psi}_2| - 3|\Psi_2|}.$$

Plot of a typical indicatrix (φ is arbitrary):



For the type D, the quartic surface splits into two ellipsoids of revolution. The space-time behaves as a uniaxial medium, and we deal with the birefringence phenomenon.

Type D

$$(\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0\,, \quad \Psi_2 \neq 0)$$

Optical metrics:

$$g_A^{p\,q} = l^p n^q + l^q n^p - \mu(m^p \bar{m}^q + m^q \bar{m}^p)\,,$$
$$g_B^{p\,q} = l^p n^q + l^q n^p - \mu^{-1}(m^p \bar{m}^q + m^q \bar{m}^p),$$
$$\mu = \frac{|1 + 2\Psi_2 - \bar{\Psi}_2| + 3|\Psi_2|}{|1 + 2\Psi_2 - \bar{\Psi}_2| - 3|\Psi_2|}.$$

l^p and n^p — two different principal null directions of the susceptibility tensor

$$\chi_{p\,q\,r\,[s} l_{t]} l^q l^r = \chi_{p\,q\,r\,[s} n_{t]} n^q n^r = 0\,.$$

Polarization vectors ($k_i \neq l_i$, $k_i \neq n_i$):

$$A: \;\; a_i = k_p \left[S(l^p n_i - n^p l_i) + m^p \bar{m}_i - \bar{m}^p m_i\right],$$
$$B: \;\; a_i = k_p \left[l^p n_i - n^p l_i + S(m^p \bar{m}_i - \bar{m}^p m_i)\right],$$

$$S = \frac{i}{\operatorname{Im}\Psi_2} \frac{|1 + 2\Psi_2 - \bar{\Psi}_2| \operatorname{Re}\Psi_2 - (1 + \operatorname{Re}\Psi_2)|\Psi_2|}{|1 + 2\Psi_2 - \bar{\Psi}_2| + 3|\Psi_2|}.$$

When Ψ_2 is real and positive, S vanishes.

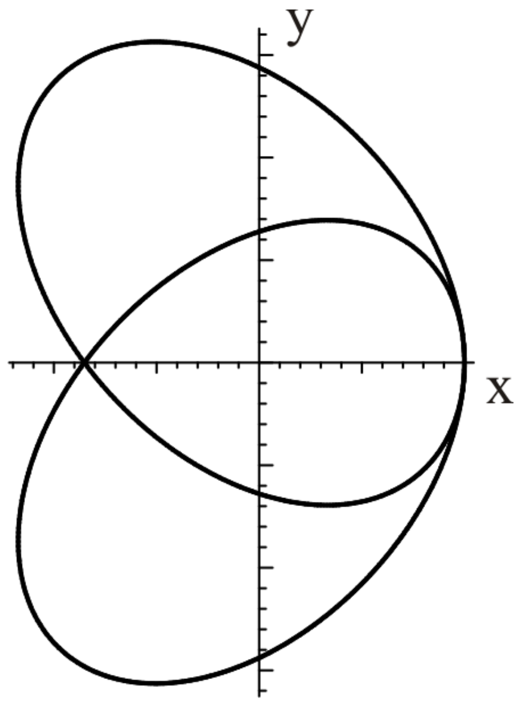
Type III

$$(\Psi_0 = \Psi_1 = \Psi_2 = 0\,, \quad \Psi_4 = 2\Psi_3^2 \neq 0)$$

Canonical form of the Fresnel equation:

$$\mathcal{G}^{(pqrs)}k_pk_qk_rk_s = \left(2k_1k_4 - 2k_2k_3 - 4|\Psi_3|^2k_1^2\right)^2 - 64|\Psi_3|^2k_1^2k_2k_3 = 0$$

Plot of a typical indicatrix (φ is arbitrary):



For the type III, the quartic surface does not split into two quadric surfaces.

Type II

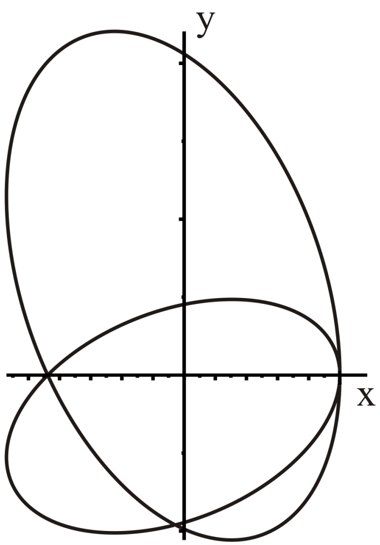
$$(\Psi_0 = \Psi_1 = 0\,, \quad 2\Psi_3^2 = \Psi_4(1 + 2\Psi_2 + 2\bar{\Psi}_2)\,, \quad \Psi_2 \neq 0)$$

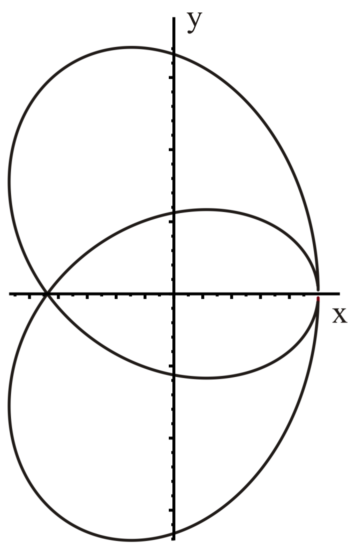
Canonical form of the Fresnel equation:

$$\mathcal{G}^{(pqrs)} k_p k_q k_r k_s = \Big\{ \Big[\big(|1{+}2\Psi_2{-}\bar{\Psi}_2|^2 {-} 9|\Psi_2|^2 \big) \left(2k_1k_4{-}2k_2k_3\right){-}4|k_1\Psi_3{-}3k_3\Psi_2|^2 \Big]^2$$

$$- 16|1{+}2\Psi_2{-}\bar{\Psi}_2|^2 |2k_1\Psi_3 - 3k_3\Psi_2|^2 k_2k_3 \Big\} (1 + 4\,\mathrm{Re}\,\Psi_2)^{-1} = 0.$$

Plot of a typical indicatrix ($\varphi = 0$ for the left panel, $\varphi = \pi/2$ for the right panel):



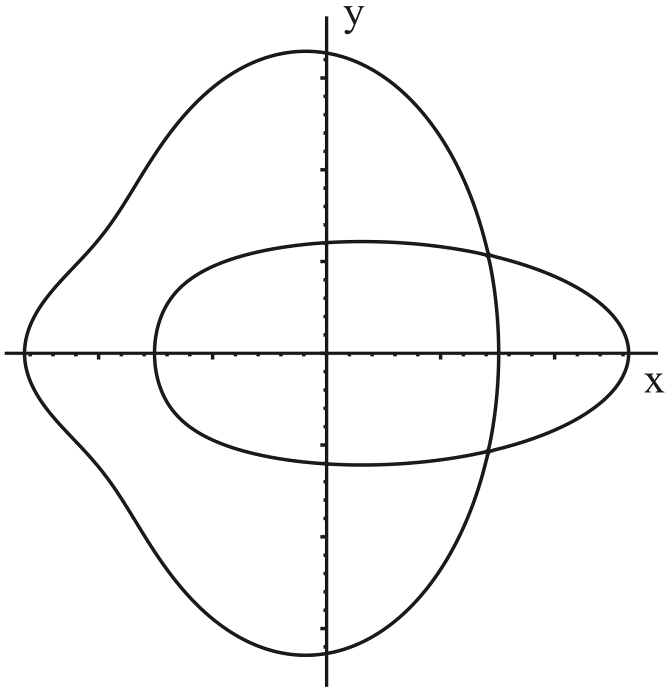


For the type II, the quartic surface does not split into two quadric surfaces.

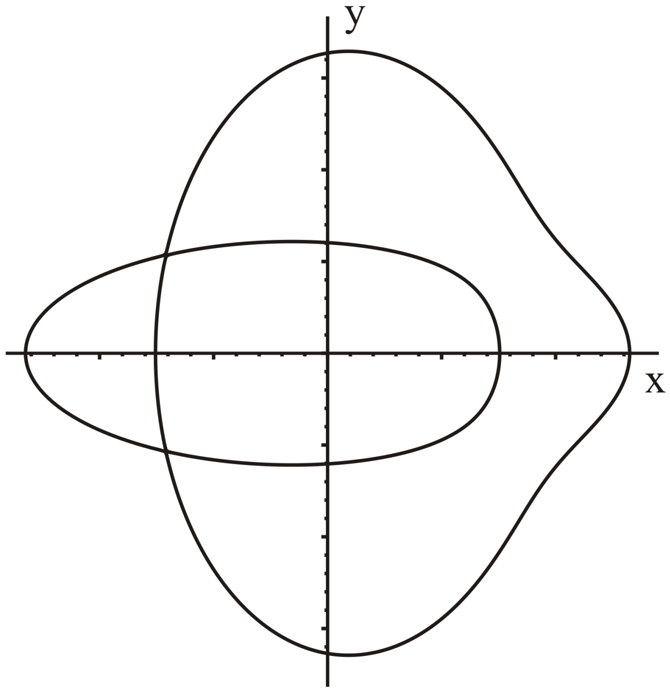
Type I

(the algebraically general type)

Plots of indicatrices ($\Psi_0 = \Psi_4 = -0.15 + 0.1i$, $\Psi_1 = \Psi_3 = 0$, $\Psi_2 = 0.15$):



$\varphi = +0.3$

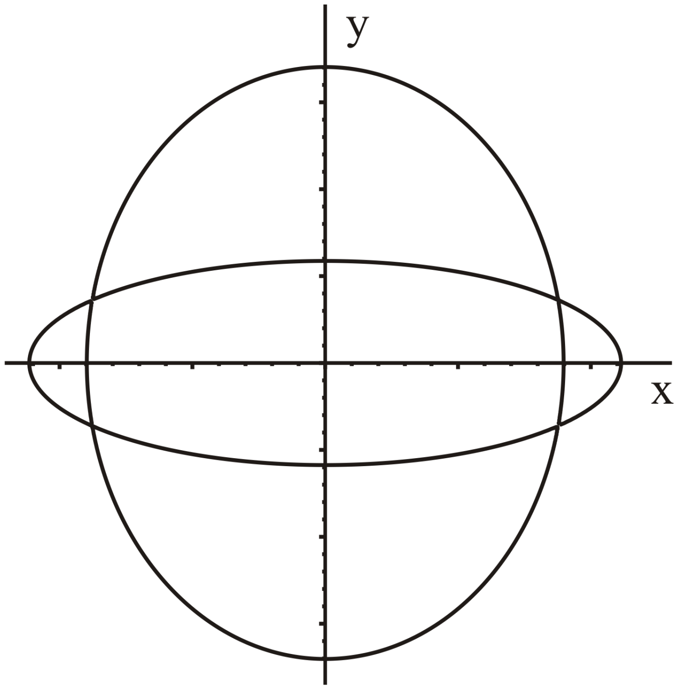


$\varphi = -0.3$

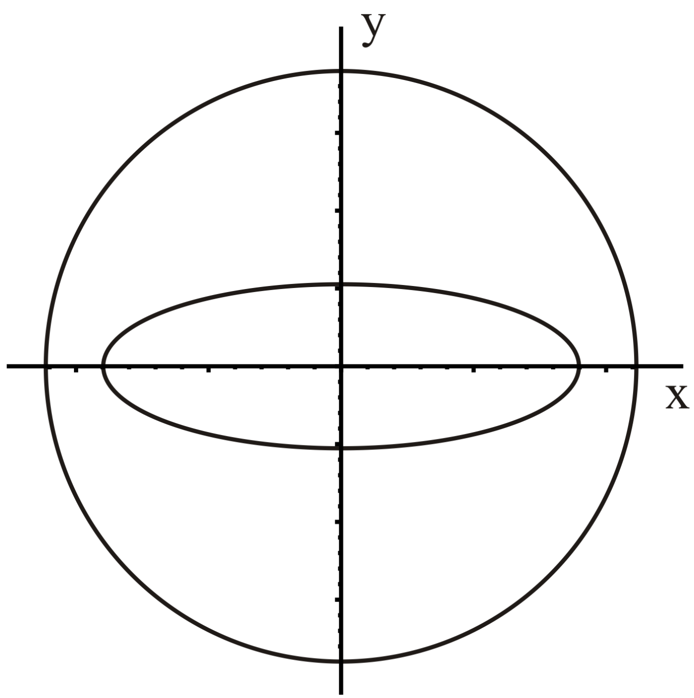
Type I

(the algebraically general type)

Plots of indicatrices ($\Psi_0 = \Psi_4 = 0.07$, $\Psi_1 = \Psi_3 = 0$, $\Psi_2 = 0.15$):



$\varphi = 0$



$\varphi = \pi/2$

Thank you for your attention!