Resurgent Transseries: Beyond (Large N) Perturbative Expansions

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Motivation: Perturbation theory is all around us, but...

- Perturbative series expansions often asymptotic ⇒ Zero radius of convergence. Why does this happen?
  - Free–theory limit usually non–analytic...
  - Singularities in the complex Borel plane: instantons, renormalons...
- Perturbative series around $z \sim \infty$,

$$F(z) \simeq \sum_{g=0}^{+\infty} \frac{F_g}{z^{g+1}}.$$

Asymptotic ⇒ Coefficients grow as $F_g \sim g!$.

- How does one make sense out of perturbation theory?

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Motivation: Borel resummation sometimes helpful, but...

- Borel transform "removes" factorial growth

$$\mathcal{B}\left[\frac{1}{z^{\alpha+1}}\right](s) = \frac{s^{\alpha}}{\Gamma(\alpha+1)}.$$

  Analytically continue $\mathcal{B}[F](s)$ throughout $s \in \mathbb{C}$.

- Borel resummation given by inverse Borel transform

$$\mathcal{S}_{\theta}F(z) = \int_0^{\mathrm{e}^{\mathrm{i}\theta}\infty} \mathrm{d}s\, \mathcal{B}[F](s)\, \mathrm{e}^{-zs}.$$

  Only defined if $\mathcal{B}[F](s)$ has no singularities along direction θ!

- How does one make sense out of perturbation theory?

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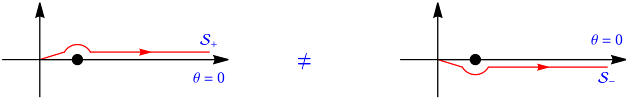
Motivation: The nature of the nonperturbative ambiguity...



- Consider lateral Borel resummations and a simple pole singularity,

$$\mathcal{B}[F](s) = \frac{1}{s - A}.$$

Find a nonperturbative ambiguity

$$\mathcal{S}_+ F - \mathcal{S}_- F = -2\pi \mathrm{i}\, \mathrm{e}^{-Az}.$$

- Perturbation theory non–Borel resummable along *any* Stokes line.
- How does one make sense out of perturbation theory?

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Motivation: The ambiguity in Quantum Mechanics...

- Ground–state energy of the quartic anharmonic oscillator [Bender–Wu]

$$F_g \sim \frac{g!}{A^g}, \qquad F_g^{(n)} \sim n\,\frac{g!}{A^g}.$$

  Also multi–instanton series suffer from nonperturbative ambiguities!
- Is the problem with perturbation theory even worse?... No: instanton ambiguities are actually the solution to defining perturbation theory!
- Ground–state energy in the double–well quartic potential [Zinn-Justin...]
  - 2–instantons ambiguity cancels perturbative ambiguity;
  - 3–instantons ambiguity cancels 1–instanton ambiguity;
  - ...

  ⇒ Ground–state energy is not only given by perturbative expansion, but rather is a sum over all multi–instanton sectors:
  ⇒ Free of any nonperturbative ambiguities!

Motivation: The ambiguity in Quantum Field Theory...

- Renormalons dominant as compared to instantons in Borel plane...
- No semiclassical description? **['t Hooft]** Recent: Yes! **[Ünsal–Dunne–Argyres...]**
- Use both (multi) renormalons and (multi) instantons to cancel all ambiguities within perturbative expansion of gauge theories...
- Define quantum field theory starting out with perturbative data and augmenting it into transseries: perturbative expansion + multi–instanton expansion + multi–renormalon expansion + $\cdots$!
- Quantum mechanical solution works generically as long as all singularities in complex Borel plane have a semiclassical description...

Motivation: The ambiguity in String Theory...

- Two–dimensional superstring theory described by the Painlevé II equation: canceling the nonperturbative ambiguity leads to median resummation of transseries! [Mariño]
- Look out: there may be more singularities in the complex Borel plane than just instantons or renormalons! Need to identify and incorporate them all in order to cancel all ambiguities (everywhere in the complex plane) and construct fully nonperturbative solutions!
- Use resurgent analysis to probe deep in the large–order behavior of perturbative expansions (around any sector) to recover all other semiclassical data ⇒ May find new (very suppressed) singularities which must also be taken into account! [Garoufalidis–Its–Kapaev–Mariño]

Outline

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Analyticity versus Non–Analyticity

- Analytic functions described by power series. But how to describe general non–analytic functions?
- Transseries augment power series with non–analytic terms, in order to describe non–analytic functions! [Écalle]
- For example with exponentials (around $x \sim 0$),

$$\exp\left(-\frac{1}{x}\right), \quad \exp\left(-\exp\left(\frac{1}{x}\right)\right), \quad \exp\left(-\exp\left(\exp\left(\frac{1}{x}\right)\right)\right), \quad \cdots$$

or with logarithms,

$$\log(x), \quad \log(\log(x)), \quad \log(\log(\log(x))), \quad \cdots$$

- Here, mainly address familiar non–analytic dependence $\mathrm{e}^{-\frac{1}{x}}$.

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Perturbative versus Nonperturbative

- One–parameter transseries (generically multi–parameters...):

$$F(z,\sigma) = \sum_{n=0}^{+\infty} \sigma^n F^{(n)}(z), \qquad F^{(n)}(z) \simeq \mathrm{e}^{-nAz} \sum_{g=1}^{+\infty} \frac{F_g^{(n)}}{z^{g+n\beta}}.$$

- Double "perturbative" expansion, both in $1/z$ and $\sigma\,\mathrm{e}^{-Az}$:
  - σ: transseries parameter... instanton–counting parameter... choice of boundary conditions... choice of integration contours... fugacity...
- Transseries: yield most general solutions to non–linear systems:
  - Feasible to solve for all unknowns $F_g^{(n)}$ of corresponding hierarchy of (non–linear but recursive) equations...
- Resurgence: coefficients $F_g^{(n)}$, $F_{g'}^{(n')}$ relate to each other!

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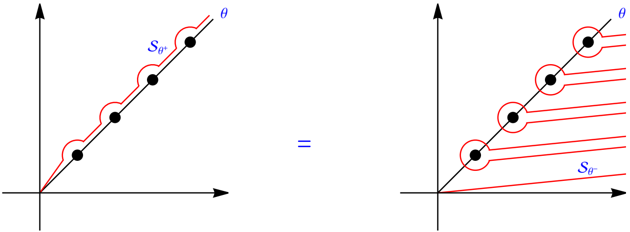
Stokes Automorphism and Ambiguities

- Singular direction θ: along which there are singularities in Borel plane.
- In original complex z–plane such direction is known as Stokes line.
- Lateral Borel resummations are related, accomplished via the Stokes automorphism, $\underline{\mathfrak{S}}_\theta$,

$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta.$$

  $\underline{\mathfrak{S}}_\theta$ codifies ambiguity!
- With $\underline{\mathfrak{S}}_\theta \equiv \mathbf{1} - \mathrm{Disc}_{\,\theta^-}$, there is an ambiguity whenever $\underline{\mathfrak{S}}_\theta \neq \mathbf{1}$.
- $\mathrm{Disc}_{\,\theta^-}$ = sum over Hankel contours encircling each singular point:



Stokes Automorphism and Perturbative Expansions

- One–parameter transseries, with single instanton action A has, nonetheless, two singular directions: $\theta = 0$ and $\theta = \pi$.
- With $F^{(n)} \equiv \mathrm{e}^{-nAz}\,\Phi_n$, Stokes automorphism along $\theta = 0$ is

$$\underline{\mathfrak{S}}_0 \Phi_n = \sum_{\ell=0}^{+\infty} \binom{n+\ell}{n} S_1^{\ell}\, \mathrm{e}^{-\ell A z}\, \Phi_{n+\ell}.$$

Stokes automorphism along $\theta = \pi$ is (sum over partitions)

$$\begin{aligned}
\underline{\mathfrak{S}}_\pi \Phi_0 &= \Phi_0, \\
\underline{\mathfrak{S}}_\pi \Phi_1 &= \Phi_1, \\
\underline{\mathfrak{S}}_\pi \Phi_2 &= \Phi_2 + S_{-1}\, \mathrm{e}^{Az}\, \Phi_1, \\
\underline{\mathfrak{S}}_\pi \Phi_3 &= \Phi_3 + 2S_{-1}\, \mathrm{e}^{Az}\, \Phi_2 + \left(S_{-2} + S_{-1}^2 \right) \mathrm{e}^{2Az}\, \Phi_1, \qquad \cdots
\end{aligned}$$

- Yields full information in terms of (possibly) infinite sequence of Stokes constants $S_\ell \in \mathbb{C}$, $\ell \in \{1, -1, -2, -3, -4, \cdots\}$.

Stokes Automorphism and Transseries

- On transseries, Stokes automorphism translates to Stokes transition

$$\underline{\mathfrak{S}}_0 F(z,\sigma) = F(z, \mathbb{S}_0(\sigma)) \quad \text{and} \quad \underline{\mathfrak{S}}_\pi F(z,\sigma) = F(z, \mathbb{S}_\pi(\sigma)),$$

with

$$\begin{aligned} \mathbb{S}_0(\sigma) &= \sigma + S_1, \\ \mathbb{S}_\pi(\sigma) &= \sigma + \sigma^2 S_{-1} + \sigma^3 \left(S_{-2} + S_{-1}^2 \right) + \sigma^4 \left(S_{-3} + \frac{5}{2} S_{-1} S_{-2} + S_{-1}^3 \right) + \cdots. \end{aligned}$$

- Transseries parameters "jump" upon Stokes transitions ⇒ Find $\mathcal{S}_+ F(z,\sigma) = \mathcal{S}_- F(z, \sigma + S_1)$ clear illustration of Stokes phenomena!
- Stokes constants control all this behavior ⇒ Can they control the cancelation of the nonperturbative ambiguity?

Outline

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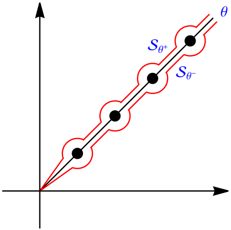
Median Resummation: Geometrical Idea



- Lateral Borel resummations

$$\mathcal{S}_{\theta^\pm} = \frac{1}{2}\left(\mathcal{S}_{\theta^+} + \mathcal{S}_{\theta^-}\right) \pm \frac{1}{2}\left(\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-}\right).$$

- θ singular direction $\Leftrightarrow$ $\underline{\mathfrak{S}}_\theta \neq \mathbf{1}$ $\Rightarrow$ ambiguity $\mathcal{S}_{\theta^+} \neq \mathcal{S}_{\theta^-}$...
- Canceling ambiguity entails $\mathcal{S}_{\theta^+} - \mathcal{S}_{\theta^-} \sim 0$ *at the level of the transseries* (also need projections $\mathbb{R}\mathrm{e}_\theta$, $\mathbb{I}\mathrm{m}_\theta$ acting on transseries parameters).

Median Resummation: Real and Imaginary Contributions

- Generically $\mathcal{S}_\theta^{\text{med}} = \mathcal{S}_{\theta^+} \circ \underline{\mathfrak{S}}_\theta^{-1/2} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta^{1/2}$... Here will focus on simpler case $\theta = 0$, real coupling, with $F_g^{(n)}$ and A real (and $\beta = 0$):
  - All Stokes constants purely imaginary $S_\ell \in \mathrm{i}\mathbb{R}$;
  - Ambiguity purely imaginary $\mathbb{I}\mathrm{m}\, F(z,\sigma) \stackrel{?}{=} 0$.
- All sectors are ambiguous for $\theta = 0$ Stokes line:

$$\left(\mathcal{S}_{0^+} - \mathcal{S}_{0^-}\right) F^{(n)}(z) = -\mathcal{S}_{0^-} \circ \left(\mathbf{1} - \underline{\mathfrak{S}}_0\right) F^{(n)}(z).$$

- With $\mathbb{R}\mathrm{e}\, F^{(n)} := \frac{1}{2}\left(\mathcal{S}_{0^+} + \mathcal{S}_{0^-}\right) F^{(n)}$ and $\mathbb{I}\mathrm{m}\, F^{(n)} := \frac{1}{2\mathrm{i}}\left(\mathcal{S}_{0^+} - \mathcal{S}_{0^-}\right) F^{(n)}$ write, for instance,

$$\mathcal{S}_{0^-} F^{(n)} = \mathbb{R}\mathrm{e}\, F^{(n)} - \mathrm{i}\, \mathbb{I}\mathrm{m}\, F^{(n)},$$

Can cancel all imaginary contributions, recursively?

Explicit Cancelation of the Ambiguity: Step 1

- Start with the perturbative sector and, for instance,

$$\mathcal{S}_{0^-} F^{(0)} = \mathbb{R}\mathrm{e}\, F^{(0)} - \mathrm{i}\, \mathbb{I}\mathrm{m}\, F^{(0)}.$$

- Via Stokes automorphism $\underline{\mathfrak{S}}_0$ can explicitly compute $\left(\mathcal{S}_{0^+} - \mathcal{S}_{0^-}\right) F^{(0)}$ as a function of $\mathcal{S}_{0^-} F^{(n)}$ $\Rightarrow$ Recursively: $\mathcal{S}_{0^-} = \mathbb{R}\mathrm{e} - \frac{1}{2}\left(\mathcal{S}_{0^+} - \mathcal{S}_{0^-}\right)$.
- Can explicitly compute the perturbative ambiguity as:

$$2\mathrm{i}\, \mathbb{I}\mathrm{m}\, F^{(0)} = S_1\, \mathbb{R}\mathrm{e}\, F^{(1)} - \frac{1}{2} S_1^3\, \mathbb{R}\mathrm{e}\, F^{(3)} + S_1^5\, \mathbb{R}\mathrm{e}\, F^{(5)} + \cdots.$$

- Need one–instanton contributions (and more) to try to cancel!

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Explicit Cancelation of the Ambiguity: Step 2

- The perturbative sector now looks like:

$$\mathcal{S}_{0^-} F^{(0)} = \mathbb{R}\mathrm{e}\, F^{(0)} - \frac{1}{2} S_1\, \mathbb{R}\mathrm{e}\, F^{(1)} + \frac{1}{4} S_1^3\, \mathbb{R}\mathrm{e}\, F^{(3)} - \frac{1}{2} S_1^5\, \mathbb{R}\mathrm{e}\, F^{(5)} + \cdots.$$

- Cancel first component of ambiguity with one–instanton contribution:

$$\mathcal{S}_{0^-} F^{(1)} = \mathbb{R}\mathrm{e}\, F^{(1)} - \mathrm{i}\, \mathbb{I}\mathrm{m}\, F^{(1)}.$$

- Still need to cancel:
  - Remaining perturbative ambiguity...
  - Full one–instanton ambiguity...

$$2\mathrm{i}\, \mathbb{I}\mathrm{m}\, F^{(1)} = 2 S_1\, \mathbb{R}\mathrm{e}\, F^{(2)} - 2 S_1^3\, \mathbb{R}\mathrm{e}\, F^{(4)} + 6 S_1^5\, \mathbb{R}\mathrm{e}\, F^{(6)} + \cdots.$$

Explicit Cancelation of the Ambiguity: Step 2

- The perturbative sector now looks like:

$$\mathcal{S}_{0^-} F^{(0)} = \mathbb{R}\mathrm{e}\, F^{(0)} - \frac{1}{2} S_1\, \mathbb{R}\mathrm{e}\, F^{(1)} + \frac{1}{4} S_1^3\, \mathbb{R}\mathrm{e}\, F^{(3)} - \frac{1}{2} S_1^5\, \mathbb{R}\mathrm{e}\, F^{(5)} + \cdots.$$

- Cancel first component of ambiguity with one–instanton contribution:

$$\frac{1}{2} S_1\, \mathcal{S}_{0^-} F^{(1)} = \frac{1}{2} S_1\, \mathbb{R}\mathrm{e}\, F^{(1)} - \frac{\mathrm{i}}{2} S_1\, \mathbb{I}\mathrm{m}\, F^{(1)}.$$

- Still need to cancel:
  - Remaining perturbative ambiguity...
  - Full one–instanton ambiguity...

$$2\mathrm{i}\, \mathbb{I}\mathrm{m}\, F^{(1)} = 2 S_1\, \mathbb{R}\mathrm{e}\, F^{(2)} - 2 S_1^3\, \mathbb{R}\mathrm{e}\, F^{(4)} + 6 S_1^5\, \mathbb{R}\mathrm{e}\, F^{(6)} + \cdots$$

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Explicit Cancelation of the Ambiguity: Step 2

- The perturbative sector now looks like:

$$\boxed{\mathcal{S}_{0^-} F^{(0)}} = \mathbb{R}\mathrm{e}\, F^{(0)} \boxed{-\frac{1}{2} S_1\, \mathbb{R}\mathrm{e}\, F^{(1)}} + \frac{1}{4} S_1^3\, \mathbb{R}\mathrm{e}\, F^{(3)} - \frac{1}{2} S_1^5\, \mathbb{R}\mathrm{e}\, F^{(5)} + \cdots.$$

- Cancel first component of ambiguity with one–instanton contribution:

$$\boxed{\frac{1}{2} S_1\, \mathcal{S}_{0^-} F^{(1)}} = \boxed{\frac{1}{2} S_1\, \mathbb{R}\mathrm{e}\, F^{(1)}} - \frac{\mathrm{i}}{2} S_1\, \mathbb{I}\mathrm{m}\, F^{(1)}.$$

- Still need to cancel:
  - Remaining perturbative ambiguity...
  - Full one–instanton ambiguity...

$$2\mathrm{i}\, \mathbb{I}\mathrm{m}\, F^{(1)} = 2 S_1\, \mathbb{R}\mathrm{e}\, F^{(2)} - 2 S_1^3\, \mathbb{R}\mathrm{e}\, F^{(4)} + 6 S_1^5\, \mathbb{R}\mathrm{e}\, F^{(6)} + \cdots$$

Explicit Cancelation of the Ambiguity: Step 3

- The perturbative and one–instanton sectors now look like:

$$\mathcal{S}_{0^-}\left(F^{(0)}+\frac{1}{2}S_1\,F^{(1)}\right) = \mathbb{R}\mathrm{e}\,F^{(0)} \boxed{+\frac{1}{4}S_1^3\,\mathbb{R}\mathrm{e}\,F^{(3)}} -\frac{1}{2}S_1^5\,\mathbb{R}\mathrm{e}\,F^{(5)}+\cdots$$
$$-\frac{1}{2}S_1^2\,\mathbb{R}\mathrm{e}\,F^{(2)}+\frac{1}{2}S_1^4\,\mathbb{R}\mathrm{e}\,F^{(4)}-\frac{3}{2}S_1^6\,\mathbb{R}\mathrm{e}\,F^{(6)}+\cdots.$$

- Contribution from the two–instantons sector:

$$\frac{1}{4}S_1^2\times\quad \mathrm{i}\,\mathbb{I}\mathrm{m}\,F^{(2)} = \boxed{\frac{3}{2}S_1\,\mathbb{R}\mathrm{e}\,F^{(3)}} -\frac{5}{2}S_1^3\,\mathbb{R}\mathrm{e}\,F^{(5)}+\frac{21}{2}S_1^5\,\mathbb{R}\mathrm{e}\,F^{(7)}+\cdots.$$

- Contribution from the three–instantons sector:

$$\frac{1}{8}S_1^3\times\quad \mathcal{S}_{0^-}F^{(3)} = \boxed{\mathbb{R}\mathrm{e}\,F^{(3)}} -\mathrm{i}\,\mathbb{I}\mathrm{m}\,F^{(3)}.$$

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Explicit Cancelation of the Ambiguity: Step ∞

- Iterating, cancel all ambiguities and construct answer:

$$\mathcal{S}_{0^-}\left(\sum_{n=0}^{+\infty}\frac{1}{2^n}S_1^n\,F^{(n)}\right)\equiv\mathcal{S}_{0^-}F\left(z,\frac{1}{2}S_1\right).$$

  $\Rightarrow$ This process constructed transseries solution!

- Starting with left Borel resummation instead, obtain $\mathcal{S}_{0^+}F\left(z,-\frac{1}{2}S_1\right)$ as expected from Stokes automorphism... But when explicitly written in terms of real contributions, would have obtained exactly the same!
- Canceled all ambiguities and obtained explicitly real answer:

$$F_{\mathbb{R}}=\mathbb{R}\mathrm{e}\,F^{(0)}-\frac{1}{4}S_1^2\,\mathbb{R}\mathrm{e}\,F^{(2)}+\frac{5}{16}S_1^4\,\mathbb{R}\mathrm{e}\,F^{(4)}+\cdots.$$

- General solution within perturbation theory!

Some Exact Results...

- Generically, for one–parameter transseries $F(z,\sigma)$ along $\theta = 0$ can compute (split $\sigma = \sigma_{\mathrm{R}} + \mathrm{i}\sigma_{\mathrm{I}}$)

$$\begin{aligned} \mathbb{I}\mathrm{m}\, F(z,\sigma) &= \left(\frac{1}{2\mathrm{i}}\, S_1 + \sigma_{\mathrm{I}}\right) \mathbb{R}\mathrm{e}\, F^{(1)} + \\ &+ \frac{1}{2\mathrm{i}} \sum_{n=2}^{+\infty} \left(\Omega(n)\, S_1^n + 2\mathrm{i} \sum_{r=0}^{[(n-1)/2]} \binom{n}{2r+1} (-1)^r \sigma_{\mathrm{R}}^{n-(2r+1)} \sigma_{\mathrm{I}}^{2r+1} + \right. \\ &+ \left. \sum_{k=1}^{n-1} \binom{n}{k} \Omega(n-k)\, S_1^{n-k} \sum_{r=0}^{[k/2]} \binom{k}{2r} (-1)^r \sigma_{\mathrm{R}}^{k-2r} \sigma_{\mathrm{I}}^{2r} \right) \mathbb{R}\mathrm{e}\, F^{(n)}, \end{aligned}$$

with $\Omega(k) = \sum_{r=1}^{k} \sum_{s=1}^{r} \binom{r}{s} (-1)^{s+1} \frac{s^k}{2^{r-1}}$.

- Can show that

$$\mathbb{I}\mathrm{m}\, F(z,\sigma) = 0$$

is satisfied to all orders by $\sigma_{\mathrm{I}} = \frac{\mathrm{i}}{2}\, S_1$ and any σ_{R}.

Some More Exact Results...

- Under previous conditions and further setting $\sigma_{\rm R} = 0$, obtain all orders median resummation

$$\mathbb{R}\mathrm{e}\, F(z,\sigma) = \mathbb{R}\mathrm{e}\, F^{(0)} + \\ + \sum_{n=1}^{+\infty} \left(\frac{1}{2^{2n}} - \sum_{k=0}^{n-1} \binom{2n}{2k+1} \frac{1}{2^{2(k+1)}}\, \Omega\left(2(n-k)-1\right) \right) S_1^{2n}\, \mathbb{R}\mathrm{e}\, F^{(2n)}.$$

- Can play the same game along $\theta = \pi$ and even set up equations to cancel ambiguities along the full real axis, $\theta = 0, \pi$... Now all Stokes constants are required, S_ℓ, $\ell \in \{1, -1, -2, -3, -4, \cdots\}$.
- Multi–parameter transseries: all Stokes constants $S_\ell^{(k)}$, $\widetilde{S}_\ell^{(k)}$ generically appear... There are now no simple closed–form expressions... But still recursively solvable... [Aniceto–RS]

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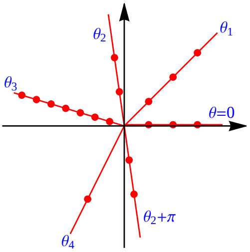
Global Picture in the Complex Borel Plane



- Global definition requires cancelation of all ambiguities...
- Cancelation only possible via explicit knowledge of the Stokes automorphism ⇒ Need to know all singularities in the complex Borel plane ⇒ Need to know all Stokes constants...
- The theory of resurgent functions allows for resummation along any direction in Borel plane ⇒ Family of sectorial analytic functions $\{\mathcal{S}_\theta F(z)\}$ ⇒ "Connect" sectorial solutions together...

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What remains to be done?

- Quantum–theoretical physical observables not only given by perturbative resummation but adequate resummations of transseries encoding full nonperturbative semiclassical data.
- Ambiguity–free via median resummation: general nonperturbative answer ... at least mathematically!
- But to implement median resummation in different physical settings:
  - Identify full set of singularities in the complex Borel plane;
  - Find physical semiclassical interpretation for each singularity.
- Question remains: How to make sure we identified all possible singularities in Borel plane? ⇒ Resurgence and large–order!

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Outline

Large–Order Dispersion Relation

- Resurgence allows to understand full asymptotic (large–order) behavior of all multi–instanton sectors.
- Large–order dispersion relation from Cauchy's theorem

$$F(z) = \frac{1}{2\pi \mathrm{i}} \int_0^{\mathrm{e}^{\mathrm{i}\theta}\cdot\infty} \mathrm{d}w\, \frac{\mathrm{Disc}_{\,\theta}\, F(w)}{w-z} - \oint_{(\infty)} \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{F(w)}{w-z}.$$

- Function $F(z)$ has branch-cut along some direction (a Stokes direction), θ in $\mathbb{C}$, and analytic elsewhere.
- Most situations show by scaling arguments that integral around infinity does not contribute [Bender–Wu] $\Rightarrow$ Cauchy's theorem provides connection between perturbative and nonperturbative expansions.

Asymptotics of (One–Parameter) Perturbative Series

- Via Stokes automorphism $\underline{\mathfrak{S}}_\theta \equiv \mathbf{1} - \mathrm{Disc}_{\theta^-}$, $F^{(0)}(z)$ has discontinuities

$$\mathrm{Disc}_0\, \Phi_0 = -\sum_{\ell=1}^{+\infty} S_1^\ell\, \mathrm{e}^{-\ell A z}\, \Phi_\ell, \qquad \mathrm{Disc}_\pi\, \Phi_0 = 0.$$

- From perturbative expansion and dispersion relation above

$$F_g^{(0)} \simeq \sum_{k=1}^{+\infty} \frac{S_1^k}{2\pi \mathrm{i}} \frac{\Gamma\left(g-k\beta\right)}{\left(kA\right)^{g-k\beta}} \sum_{h=1}^{+\infty} \frac{\Gamma\left(g-k\beta-h+1\right)}{\Gamma\left(g-k\beta\right)} F_h^{(k)} \left(kA\right)^{h-1},$$

here used asymptotic expansions for multi–instanton contributions.

- Instructive to explicitly write down first terms in double–series,

$$\begin{aligned} F_g^{(0)} \simeq\ & \frac{S_1}{2\pi \mathrm{i}} \frac{\Gamma\left(g-\beta\right)}{A^{g-\beta}} \left(F_1^{(1)} + \frac{A}{g-\beta-1} F_2^{(1)} + \cdots \right) + \\ & + \frac{S_1^2}{2\pi \mathrm{i}} \frac{\Gamma\left(g-2\beta\right)}{\left(2A\right)^{g-2\beta}} \left(F_1^{(2)} + \frac{2A}{g-2\beta-1} F_2^{(2)} + \cdots \right) + \cdots \end{aligned}$$

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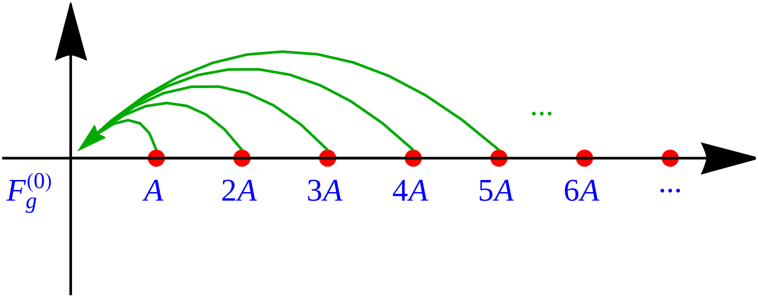
Asymptotics of (One–Parameter) Perturbative Series



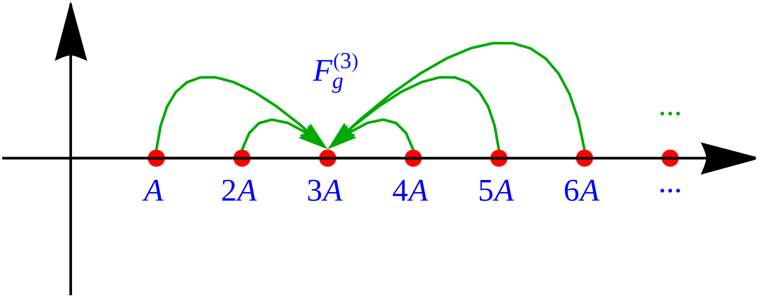
Asymptotics of (One–Parameter) Multi–Instanton Series

- Stokes automorphism yields discontinuities $\mathrm{Disc}_0\, \Phi_n$ and $\mathrm{Disc}_\pi\, \Phi_n$.
- From perturbative expansion and dispersion relation

$$F_g^{(n)} \simeq \sum_{k=1}^{+\infty} \binom{n+k}{n} \frac{S_1^k}{2\pi\mathrm{i}} \frac{\Gamma\left(g-k\beta\right)}{\left(kA\right)^{g-k\beta}} \sum_{h=1}^{+\infty} \frac{\Gamma\left(g-k\beta-h\right)}{\Gamma\left(g-k\beta\right)} F_h^{(n+k)} \left(kA\right)^h + \\ + \frac{S_{-1}}{2\pi\mathrm{i}} \left(n-1\right) \frac{\Gamma\left(g+\beta\right)}{\left(-A\right)^{g+\beta}} \sum_{h=1}^{+\infty} \frac{\Gamma\left(g+\beta-h\right)}{\Gamma\left(g+\beta\right)} F_h^{(n-1)} \left(-A\right)^h + \cdots$$

- Relates coefficients of perturbative expansion @ n–instanton sector with sums over coefficients of perturbative expansions @ *all* other multi–instanton sectors.
- All Stokes factors now needed ⇒ Analysis essentially boiled down asymptotic problem to computing these Stokes factors!

Asymptotics of (One–Parameter) Multi–Instanton Series



Asymptotics of (Two–Parameter) Perturbative Series

- Two–parameters transseries with instanton actions A and $-A$...
- Already at perturbative level Stokes automorphism yields discontinuities $\mathrm{Disc}_0\, \Phi_{(0|0)}$ and $\mathrm{Disc}_\pi\, \Phi_{(0|0)}$.
- From perturbative expansion and dispersion relation

$$F_g^{(0|0)} \simeq \sum_{k=1}^{+\infty} \frac{\left(S_1^{(0)}\right)^k}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta_{k0})}{(kA)^{g-\beta_{k0}}} \sum_{h=1}^{+\infty} \frac{\Gamma(g-\beta_{k0}-h+1)}{\Gamma(g-\beta_{k0})} F_h^{(k|0)} (kA)^{h-1} + \\ + \sum_{k=1}^{+\infty} \frac{\left(\widetilde{S}_{-1}^{(0)}\right)^k}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta_{0k})}{(-kA)^{g-\beta_{0k}}} \sum_{h=1}^{+\infty} \frac{\Gamma(g-\beta_{0k}-h+1)}{\Gamma(g-\beta_{0k})} F_h^{(0|k)} (-kA)^{h-1}.$$

- Leading asymptotics now clearly distinct! Schematically:

$$F_g^{(0|0)} \simeq \frac{S_1^{(0)}}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta)}{A^{g-\beta}} F_1^{(1|0)} + \frac{\widetilde{S}_{-1}^{(0)}}{2\pi\mathrm{i}} \frac{\Gamma(g-\beta)}{(-A)^{g-\beta}} F_1^{(0|1)} + \cdots$$

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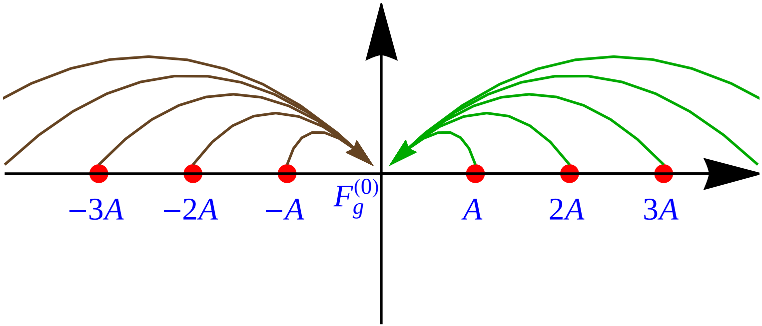
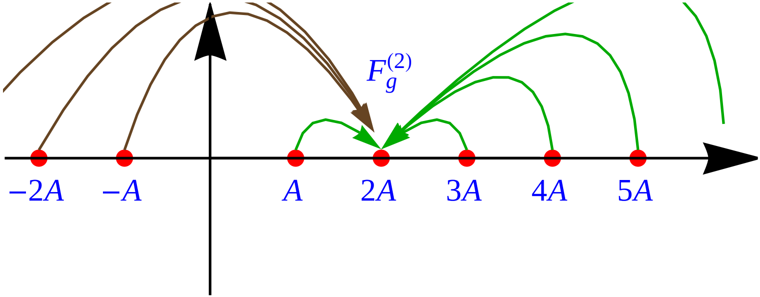
Asymptotics of (Two–Parameter) Perturbative Series



Asymptotics of (Two–Parameter) Multi–Instanton Series



Asymptotics of (Two–Parameter) Multi–Instanton Series

- New, distinct, visible, features in the multi–instantonic asymptotics:
  - Intricate asymptotics depending on many Stokes constants $S_\ell^{(k)}$, $\widetilde{S}_\ell^{(k)}$.
  - In addition to familiar $g!$ growth of large–order coefficients find new large–order growth of type $g! \log g$: dominant!
- Precisely these asymptotics found in many examples!
  - Painlevé I equation (2d gravity) [Garoufalidis–Its–Kapaev–Mariño]
  - Painlevé I equation (2d gravity) [Aniceto–RS–Vonk]
  - One–cut quartic matrix model [Aniceto–RS–Vonk]
  - Painlevé II equation (2d supergravity) [RS–Vaz]
  - Two–cuts quartic matrix model [RS–Vaz]
  - Quantum mechanics with elliptic–type potential [Başar–Dunne–Ünsal]

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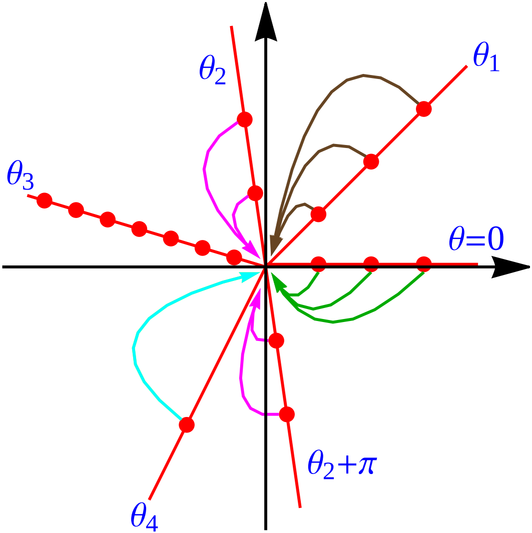
Asymptotics of Multi–Parameter Multi–Instanton Series



Outline

Random Matrices and 't Hooft Large N Limit

- Large N limit without renormalons?... Matrix models!
- Hermitian one–matrix model with polynomial potential $V(z)$,

$$Z = \frac{1}{\mathrm{vol}(\mathrm{U}(N))} \int \mathrm{d}M \, \exp\left(-\frac{1}{g_s} \mathrm{Tr}\, V(M)\right).$$

- Consider limit $N \to +\infty$ while $t = g_s N$ fixed ['t Hooft]. In this case free energy $F = \log Z$ has perturbative genus expansion,

$$F \simeq \sum_{g=0}^{+\infty} F_g(t)\, g_s^{2g-2}.$$

- Large–order behavior $F_g \sim (2g)!$ renders topological genus expansion as asymptotic approximation [Shenker].

Motivation from Gaussian Example: $V(z) = \frac{1}{2}z^2$

- Saddle–point of Z described by spectral curve $y^2 = x^2 - 4t$, asymptotic expansion as $N \to +\infty$ yields:

$$F_g(t) = \frac{B_{2g}}{2g\,(2g-2)}\, t^{2-2g}, \qquad g \geq 2.$$

- Gaussian example simple enough to obtain closed form solution,

$$F(g_s, N) = \frac{1}{2} N \left(N \log g_s - \log 2\pi \right) + \log G_2(N+1).$$

- How can one recover exact solution from asymptotic expansion?

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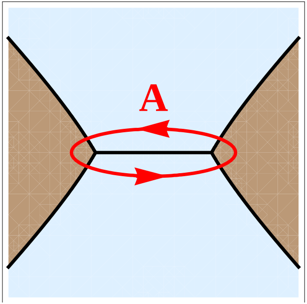
All Multi–Instanton Sectors from A–Cycles

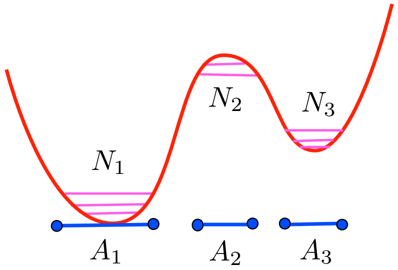
- Considering all multi–instanton sectors

$$F^{(n)} = \frac{\mathrm{i}}{g_s}\left(\frac{t}{n} + \frac{g_s}{2\pi n^2}\right) \times \exp\left(-n\,\frac{A(t)}{g_s}\right),$$

where instanton action is $A(t) = 2\pi i t \Rightarrow$ One–parameter transseries!



Interacting Theory: Quartic Matrix Model



- Potential $V(z) = \frac{1}{2}z^2 - \frac{\lambda}{24}z^4$ generically three–cuts solution.
- One–cut solution $y^2 = \left(1 - \frac{\lambda}{6}\left(z^2 + 2\alpha^2\right)\right)^2 \left(z^2 - 4\alpha^2\right)$.
- Two–cuts $\mathbb{Z}_2$–symmetric solution $y^2 = \frac{1}{36}\lambda^2 z^2 \left(z^2 - a^2\right)\left(z^2 - b^2\right)$.

Resurgent Solution around One–Cut Background

- Transseries solution to (quartic) string equation:

$$\mathcal{R}(x)\left\{1-\frac{\lambda}{6}\Big(\mathcal{R}(x-g_s)+\mathcal{R}(x)+\mathcal{R}(x+g_s)\Big)\right\}=x.$$

- Requires both "instanton" actions $+A$ and $-A$, leading to transseries:

$$\mathcal{R}(x,\sigma_1,\sigma_2)=\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}\sigma_1^n\sigma_2^m\,\mathrm{e}^{-(n-m)A(x)/g_s}\sum_{g=\beta_{nm}}^{+\infty}g_s^g\,R_g^{(n|m)}(x).$$

- Fully nonperturbative solution ⇒ Via Stokes transitions can move anywhere in (multi–cut) phase diagram.
- Extensive resurgent checks of large–order asymptotics on both perturbative and multi–instantonic sectors!

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Double–Scaling Limit and the Painlevé I Equation

- DSL yields Painlevé I equation for $u(z) = -F''_{\rm ds}(z)$

$$u^2(z) - \frac{1}{6} u''(z) = z.$$

- Perturbative solution

$$u(z) \simeq \sqrt{z} \sum_{g=0}^{+\infty} \frac{u_g}{z^{\frac{5}{2}g}},$$

yields recursion equation; obtain asymptotic expansion

$$u(z) \simeq \sqrt{z} \left(1 - \frac{1}{48} z^{-\frac{5}{2}} - \frac{49}{4608} z^{-5} + \cdots \right).$$

- A second order differential equation ⇒ Yields two instanton actions

$$A = \pm \frac{8\sqrt{3}}{5}.$$

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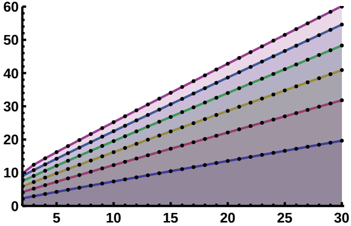
Two–Parameters Transseries Solution

- General two–parameters transseries solution is ($g_s = z^{-5/4}$):

$$u(g_s, \sigma_1, \sigma_2) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m \, \mathrm{e}^{-(n-m)\frac{A}{g_s}} \left(\sum_{k=0}^{\min(n,m)} \log^k(g_s) \cdot \Phi_{(n|m)}^{[k]}(g_s) \right).$$

- Checked nonperturbative sectors via resurgent large–order analysis.
- Resurgence allows extremely accurate tests: at genus $g = 30$, including six instantons corrections, results correct up to 60 decimal places!



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Description of Nonperturbative Sectors

- Properties of $\Phi_{(n|m)}$ nonperturbative sectors

$$\Phi_{(n|m)}(g_s) = \sum_{k=0}^{\min(n,m)} \log^k(g_s) \cdot \Phi_{(n|m)}^{[k]}(g_s).$$

  - $\Phi_{(n|m)}$, $n \neq m$ $\Rightarrow$ generically has expansion in g_s.
  - $\Phi_{(n|n)}$ $\Rightarrow$ has expansion in g_s^2 and no logarithms.
  - $\Phi_{(n|0)}$ and $\Phi_{(0|n)}$ $\Rightarrow$ no logarithms.
  - Logarithm sectors not independent: $\Phi_{(n|m)}^{[k]} = \frac{1}{k!}\left(\frac{4(m-n)}{\sqrt{3}}\right)^k \Phi_{(n-k|m-k)}^{[0]}$.
  - Physical instanton series $\Phi_{(n|0)}$ as disk amplitudes of ZZ–branes.
  - Full interpretation of "generalized" instanton series open...

Stokes Constants for the Quartic Matrix Model

- Stokes constant $S_1^{(0)}$ computed from first principles (one–loop around one–instanton) in matrix model and DSL, $S_1^{(0)} = -\mathrm{i}\,\frac{3^{1/4}}{2\sqrt{\pi}}$ [David].
- All other Stokes constants $S_\ell^{(k)}$, $\widetilde{S}_\ell^{(k)}$ so far only computed numerically $\Rightarrow$ Require extra physical input!

		Precision	From	Order
$S_1^{(0)}$	$-0.371257624642845568\ldots\,\mathrm{i}$	∞	$\Phi_{(0\vert 0)}^{[0]}$	1^{-g}
$S_2^{(0)}$	$0.500000000000000000\ldots\,\mathrm{i}$	20	$\Phi_{(1\vert 0)}^{[0]}$	2^{-g}
$S_3^{(0)}$	$-0.897849124725732240\ldots\,\mathrm{i}$	13	$\Phi_{(2\vert 0)}^{[0]}$	3^{-g}
$S_1^{(1)}$	$-4.879253817220057751\ldots\,\mathrm{i}$	81	$\Phi_{(1\vert 1)}^{[0]}$	1^{-g}
$S_2^{(1)}$	$9.856875980487862735\ldots\,\mathrm{i}$	19	$\Phi_{(2\vert 1)}^{[0]}$	2^{-g}
$S_1^{(2)}$	$-22.825711248125715287\ldots\,\mathrm{i}$	36	$\Phi_{(2\vert 2)}^{[0]}$	1^{-g}
$\widetilde{S}_1^{(2)}$	$2.439626908610028875\ldots\,\mathrm{i}$	112	$\Phi_{(2\vert 0)}^{[0]}$	1^{-g}
$\widetilde{S}_1^{(3)}$	$15.217140832083810191\ldots\,\mathrm{i}$	108	$\Phi_{(3\vert 1)}^{[0]}$	1^{-g}
$\widetilde{S}_1^{(4)}$	$45.334204678679729580\ldots\,\mathrm{i}$	108	$\Phi_{(4\vert 2)}^{[0]}$	1^{-g}

- Many relations (reality constraints) between these constants!

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Transseries Framework: General Picture

- Transseries *ansatz* for resurgent function,

$$F(\boldsymbol{\sigma}, g_s) = \sum_{\mathbf{n} \in \mathbb{N}^k} \boldsymbol{\sigma}^{\mathbf{n}} \, \mathrm{e}^{-\frac{\mathbf{n} \cdot \mathbf{A}}{g_s}} \, \Phi_{(\mathbf{n})}(g_s).$$

- For matrix models, minimal/topological strings:
  - "Generalized" instanton sectors labeled by $\mathbf{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$.
  - $\mathbf{n} = (0, \ldots, 0)$ sector usual perturbative sector.
  - $\mathbf{n} = (n, 0, \ldots, 0)$ sector usual multi–instanton sector.
  - Expansions $\Phi_{(\mathbf{n})}$ include asymptotic series and logarithms.
  - Generically $A_i \in \mathbb{C} \Rightarrow$ Many new sectors!
  - Sectors $n_i \neq n_j$, $\forall_{i,j} \Rightarrow$ Generically $\Phi_{(\mathbf{n})}$ has expansion in g_s.
  - Sectors with $\mathbf{n} \cdot \mathbf{A} = 0 \Rightarrow$ Generically $\Phi_{(\mathbf{n})}$ has expansion in g_s^2.

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Outline

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Holomorphic Effective Potential

- What exactly controls saddle–points/asymptotics of matrix integral?
- In diagonal gauge, $M = \mathrm{diag}\left(\lambda_1, \ldots, \lambda_N\right)$, holomorphic effective potential

$$V_{\mathrm{h;eff}}(\lambda) = \int_a^\lambda \mathrm{d}z\, y(z)$$

appears at leading order in large N expansion of the matrix integral

$$Z \sim \int \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \exp\left(-\frac{1}{g_s} \sum_{i=1}^{N} V_{\mathrm{h;eff}}(\lambda_i) + \cdots\right).$$

- Multi–dimensional ordinary integral... but hard to evaluate explicitly!
- Zero–locus $\mathcal{V}_0 = \left\{ z \in \mathbb{C} \,|\, \mathbb{R}\mathrm{e}\, V_{\mathrm{h;eff}}(z) = 0 \right\}$ constructs “spectral network” suitable for (non–linear) steepest–descent analysis.

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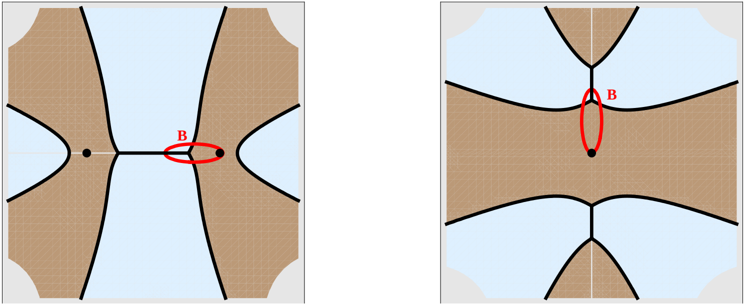
Multi–Instanton Sectors from B–Cycles



- Instantons from B–cycles [David, Seiberg–Shih, Mariño–RS–Weiss, RS–Vaz].
- Stokes lines ("jumps" in Borel plane): $\mathbb{Im}\left(\frac{A(t)}{g_s}\right) = 0$.
- Anti–Stokes lines (phase boundaries): $\mathbb{Re}\left(\frac{A(t)}{g_s}\right) = 0$.

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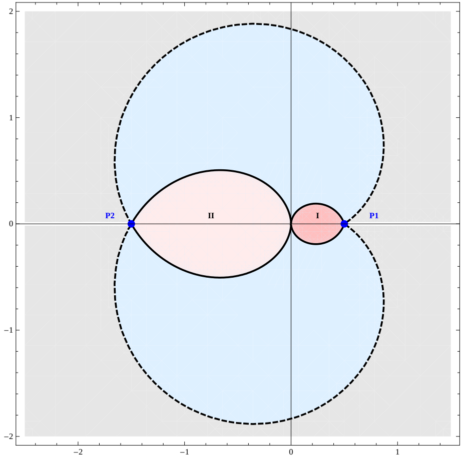
Quartic Phase Diagram for Complex 't Hooft Coupling



- Three–cuts anti–Stokes phase **[Eynard–Mariño, Mariño–Putrov–Pasquetti, Aniceto–RS–Vonk]**.
- Trivalent–tree phase **[David, Bertola, RS–Vaz]**.

Partition Function: Grand–Canonical & Transseries

- Anti–Stokes phase with 3 cuts, characterized by fillings N_1, N_2, N_3.
- Grand–canonical partition function as sum over all possible arrangements of eigenvalues across cuts [Bonnet–David–Eynard]

$$\mathcal{Z}(\zeta_1, \zeta_2, \zeta_3) = \sum_{N_1+N_2+N_3=N} \zeta_1^{N_1} \zeta_2^{N_2} \zeta_3^{N_3}\, Z(N_1, N_2, N_3).$$

- In Stokes and anti–Stokes regions: dominant canonical configuration $\mathbb{Z}_2$ symmetric $\Rightarrow$ Reference configuration $N_1^\star = N_3^\star$ and $N_2^\star = N - 2N_1^\star$,

$$\begin{aligned}\mathcal{Z}(\sigma_1, \sigma_2) &= \sum_{n=-2N_1^\star}^{N_2^\star} \sum_{m=-N_1^\star}^{N_1^\star+n} \sigma_1^n \sigma_2^m\, Z(N_1^\star + m, N_2^\star - n, N_1^\star + n - m) \\ &= Z(N_1^\star, N_2^\star, N_1^\star) \sum_{n=-2N_1^\star}^{N_2^\star} \sum_{m=-N_1^\star}^{N_1^\star+n} \sigma_1^n \sigma_2^m\, Z^{(n,m)}.\end{aligned}$$

Characterization of Multi–Instanton Sectors

- Change variables $t = t_1 + t_2 + t_3$, $s = -t_1 - t_3$, $u = t_1 - t_3$

$$Z^{(n,m)} = q^{\frac{n^2}{2}} \widetilde{q}^{\frac{(n+2m)^2}{2}} \exp\left(-\frac{nA}{g_s}\right) \left\{1 + \mathcal{O}(g_s)\right\},$$

$$A = \partial_s F_0, \qquad q = \exp\left(\partial_s^2 F_0\right), \qquad \widetilde{q} = \exp\left(\partial_u^2 F_0\right).$$

- At large $N_1^\star$, $N_2^\star$ extend sum from $-\infty$ to $+\infty$, exchange sum over n, m with g_s expansion, and write $\mathcal{Z}$ in terms of Jacobi theta functions:

$$\vartheta_2\left(z\,|\,q\right) = \sum_{n\in\mathbb{Z}} q^{\left(n+\frac{1}{2}\right)^2} z^{n+\frac{1}{2}}, \qquad \vartheta_3\left(z\,|\,q\right) = \sum_{n\in\mathbb{Z}} q^{n^2} z^n.$$

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Multi–Cuts, Multi–Instantons and Theta Functions

- Partition function at order g_s^0 (with $z = \frac{\sigma_1^2}{\sigma_2}\, \mathrm{e}^{-\frac{2A}{g_s}}$ and $\widetilde{z} = \sigma_2$)

$$Z_0 = Z_0^\star \left(\vartheta_2 \left(z \,|\, q^2 \right) \vartheta_2 \left(\widetilde{z} \,|\, \widetilde{q}^2 \right) + \vartheta_3 \left(z \,|\, q^2 \right) \vartheta_3 \left(\widetilde{z} \,|\, \widetilde{q}^2 \right) \right).$$

- Free energy from Jacobi triple product

$$\log \vartheta_3 \left(z \,|\, q \right) = -\frac{1}{12} \log q + \log \eta(q) + \sum_{k=1}^{+\infty} \frac{(-1)^k}{k} \frac{z^k + z^{-k}}{q^k - q^{-k}},$$

with $\eta(q)$ Dedekin's eta function.

- Distinct reference backgrounds (described by distinct instanton sectors) will be either exponentially suppressed or exponentially enhanced, with respect to reference configuration...

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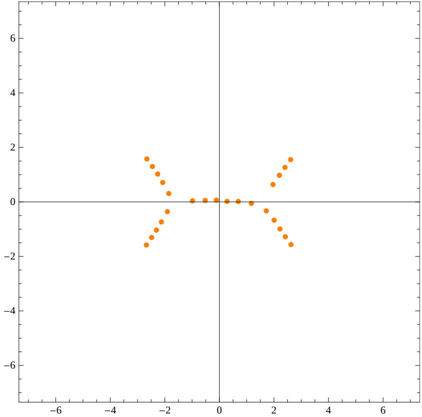
Trivalent Phase Numerics $N = 25$, $t = 5$



Transseries construction? Borel singularities and instantons? Asymptotics more stringy–like rather than theta–like? Strong 't Hooft coupling?

Outline

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Matrix Models as Strings on Calabi–Yau Geometries

- Topological string B–model on local Calabi–Yau geometries has large N duality to matrix models [Dijkgraaf–Vafa]. Also true for generic mirrors of toric geometries [Mariño, Bouchard–Klemm–Mariño–Pasquetti].
- Non–trivial information about this six–dimensional CY geometry: encoded in Riemann surface ⇔ spectral curve of matrix model.
- In Stokes regions large N duality yields B–model closed strings ⇒ Will closed string theory preserve the overall picture?
- Start by considering simpler examples: B–model on mirrors of $\mathbb{K}_{\mathbb{P}^2} = \mathcal{O}(-3) \to \mathbb{P}^2$ and $\mathbb{K}_{\mathbb{P}^1 \times \mathbb{P}^1} = \mathcal{O}(-2,-2) \to \mathbb{P}^1 \times \mathbb{P}^1$ (fibrations over curves) ⇒ Fully solved perturbatively [Haghighat–Klemm–Rauch].

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Perturbative Holomorphic Anomaly Equations

- Holomorphic anomaly equations [Bershadsky–Cecotti–Ooguri–Vafa]

$$\frac{\partial F_g^{(0)}}{\partial S^{ij}} = \frac{1}{2}\left(D_i D_j F_{g-1}^{(0)} + \sum_{h=1}^{g-1} \partial_i F_{g-h}^{(0)} \partial_j F_h^{(0)} \right), \qquad g \geq 2.$$

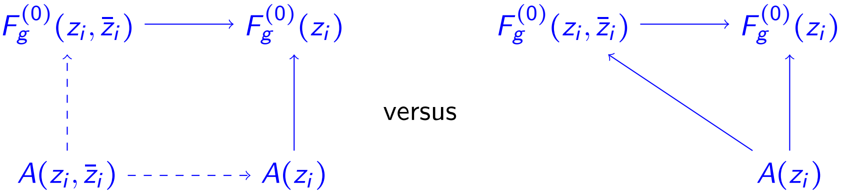
  Here D_i covariant derivative in complex structure moduli space (holomorphic dependence); S^{ij} propagators or "potentials" for Yukawa couplings (also anti–holomorphic dependence).

- Perturbative solution is polynomial in propagators [Yamaguchi–Yau, Alim–Lange–Mayr]. For local CY only need S^{ij},

$$F_g^{(0)} = \mathrm{Pol}\left(S^{ij}; 3g-3\right), \qquad g \geq 2.$$

- $F_g^{(0)}(z_i, \bar{z}_i)$ depends on holomorphic and anti–holomorphic complex structure moduli ⇒ What is large–order behavior?

Nonperturbative Holomorphic Anomaly Equations?



- Rewrite holomorphic anomaly equations for partition function $Z \Rightarrow$ Naturally solved with transseries *ansatz*:

$$Z = \exp\left(\sum_{\mathbf{n}} \boldsymbol{\sigma}^{\mathbf{n}} F^{(\mathbf{n})}\right).$$

Nonperturbative Holomorphic Anomaly Equations

- Complex structure moduli space of dimension one (single holomorphic coordinate z and single propagator S); one–parameter transseries:
  - Instanton action is holomorphic: $\partial_S A = 0$.
  - Nonperturbative version of holomorphic anomaly equations ($A^{(n)} \equiv nA$):

$$\left(\partial_S - \frac{1}{2}\left(\partial_z A^{(n)}\right)^2\right) F_g^{(n)} = -\sum_{h=1}^{g} \mathcal{D}_h^{(n)} F_{g-h}^{(n)} + \\ + \frac{1}{2} \sum_{m=1}^{n-1} \sum_{h=0}^{g-1} \left(\partial_z F_{h-1}^{(m)} - \partial_z A^{(m)} F_h^{(m)}\right)\left(\partial_z F_{g-2-h}^{(n-m)} - \partial_z A^{(n-m)} F_{g-1-h}^{(n-m)}\right).$$

- ...Fully generalizable to multi–parameter transseries and multi–dimensional complex moduli spaces... [Couso–Edelstein–RS–Vonk]

Transseries Solution of Holomorphic Anomaly Equations

- Multi–instanton free energies have the form

$$F_g^{(n)} = \sum_{\{\gamma_n\}} \mathrm{e}^{\frac{1}{2}\, a(n;\gamma_n) (\partial_z A)^2 S}\, \mathrm{Pol}\left(S; 3\left(g+1-\lambda\left(n;\gamma_n\right)\right)\right).$$

- Depend on purely combinatorial data $\{a, \lambda, \gamma\}$ encoded in generating function

$$\Phi = \prod_{m=1}^{+\infty} \frac{1}{1-\varphi\, E^{m^2}\, \rho^m} = \sum_{n=0}^{+\infty} \rho^n \sum_{\{\gamma_n\}} E^{a(n;\gamma_n)}\, \varphi^{\lambda(n;\gamma_n)} \left(1+\mathcal{O}(\varphi)\right).$$

- One and two instanton examples:

$$\begin{aligned} F_g^{(1)} &= \mathrm{e}^{\frac{1}{2}(\partial_z A)^2 S}\, \mathrm{Pol}\left(S; 3g\right), \\ F_g^{(2)} &= \mathrm{e}^{\frac{1}{2}\, 2 (\partial_z A)^2 S}\, \mathrm{Pol}\left(S; 3(g+1-2)\right) + \\ &\quad + \mathrm{e}^{\frac{1}{2}\, 4 (\partial_z A)^2 S}\, \mathrm{Pol}\left(S; 3(g+1-1)\right). \end{aligned}$$

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Resurgent Properites of Closed String Transseries

- Instanton action holomorphic ⇒ Can *still* compute A as appropriate combinations of periods in the geometry [Drukker–Mariño–Putrov].
- Calculate multi–instanton sectors $F_g^{(n)}(z,S)$ whose large–order behavior matches resurgent predictions ⇒ In particular "anti–holomorphic large–order growth" is mild (sub–leading).
- Holomorphic ambiguities may be fixed at conifold points (or else from resurgence!) ⇒ Full construction shows nonperturbative integrability of the holomorphic anomaly equations!

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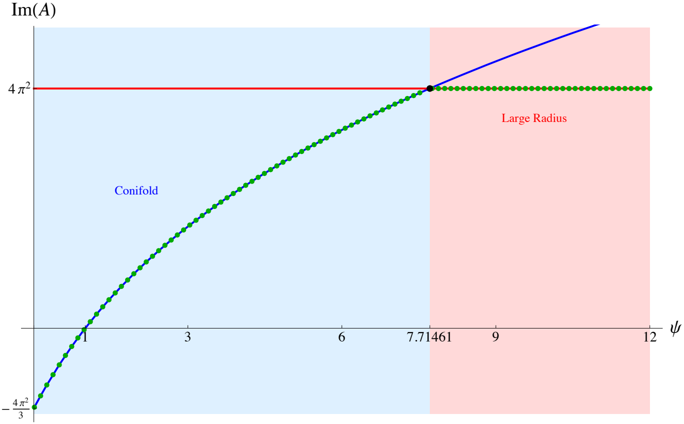
Local $\mathbb{P}^2$: Checks of Instanton Action(s)



Outline

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Summary and Future Directions

- Wrap–up:
  - Observables described by resurgent functions/transseries: median resummation cancels all ambiguities ⇒ Define observables nonperturbatively starting out from perturbation theory.
  - Constructed and rigorously tested resurgent solutions in many models, going beyond perturbative large N expansion ⇒ Rich phase diagram ⇒ Holographically dual description within (stringy) Stokes phase.
- Upcoming:
  - Fully describe generalized instanton sectors: Stokes constants?
  - Deal with trivalent–tree phase: what are its asymptotics?
  - What is (grand–canonical) transseries partition function?
  - Interesting to extend to multi–matrix models and QFT and gauge theories... Need fully general results on resurgent transseries?

Based on work in collaboration with:

- Inês Aniceto, Ricardo Couso, José Edelstein, Marcos Mariño, Sara Pasquetti, Ricardo Vaz, Marcel Vonk, Marlene Weiss, 0711.1954, 0809.2619, 0907.4082, 1106.5922, 1302.5138, 1308.1115, 1308.1695.
- I. Aniceto, R. Couso, J. Edelstein, R. Vaz, M. Vonk, arXiv: Upcoming...

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