Analysis of the Blandford-Znajek power for a rapidly rotating black hole

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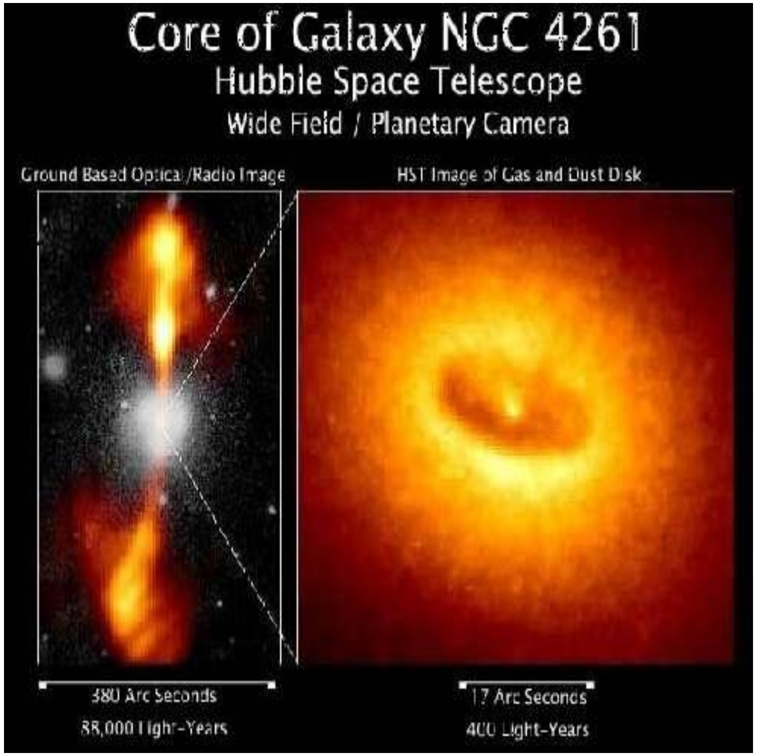
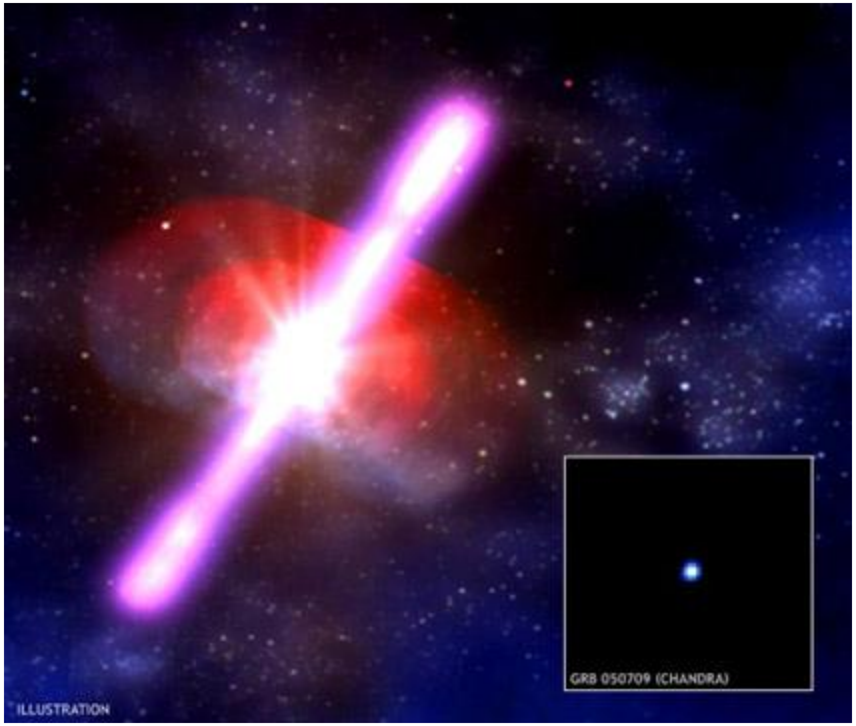
Introduction

Active Galactic Nuclei(AGN)



Gamma-Ray Burst(GRB)



"Black Hole(BH)" works as a central engine!

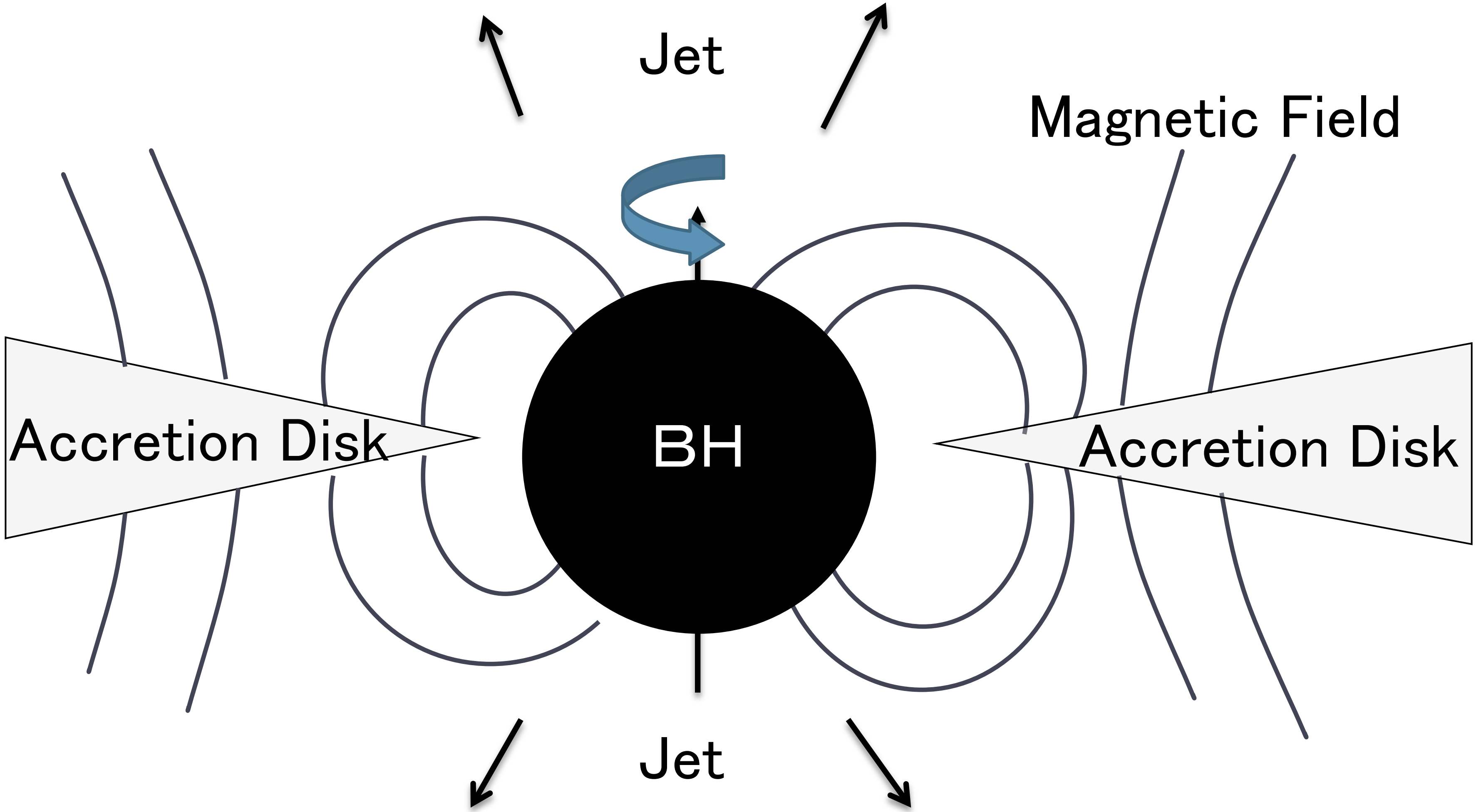
How to generate the energy

- Mass accretion on to the black hole
- Rotational energy of the black hole
  - ▶Penrose process
  - ▶Super-radiance
  - ▶Blandford-Znajek(BZ) mechanism (Blandford & Znajek 1977)

"An outward Poynting flux can exist on the event horizon of a rotating BH when plasma fills around the BH."

⇒ The rotational energy of BH is extracted by magnetic fields. In particular, the BZ mechanism is expected as an energy source of a relativistic jet.

Black hole magnetosphere with plasma



Realistic astrophysical situation is complicated.

Stationary BH magnetosphere in the force–free system

To make the problem simple, let us consider the following system:

- Kerr background geometry.
- Stationary and axisymmetric electromagnetic field.
- Strong magnetic field, that is, ignoring the inertia of plasma = the Lorentz force is zero.
  (the force–free approximation)

Kerr back ground geometry

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi + \frac{A\sin^2\theta}{\Sigma}d\phi^2 \\
&\quad + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2
\end{aligned}
$$

$$
\begin{aligned}
\Sigma &= r^2 + a^2\cos^2\theta \\
\Delta &= r^2 - 2Mr + a^2 \\
A &= (r^2+a^2)^2 - a^2\Delta\sin^2\theta
\end{aligned}
$$

- M: mass parameter. a : spin parameter.
- To express a BH geometry, $0 \le |a| \le M$
- The location of the event horizon is determined by $\Delta = 0$

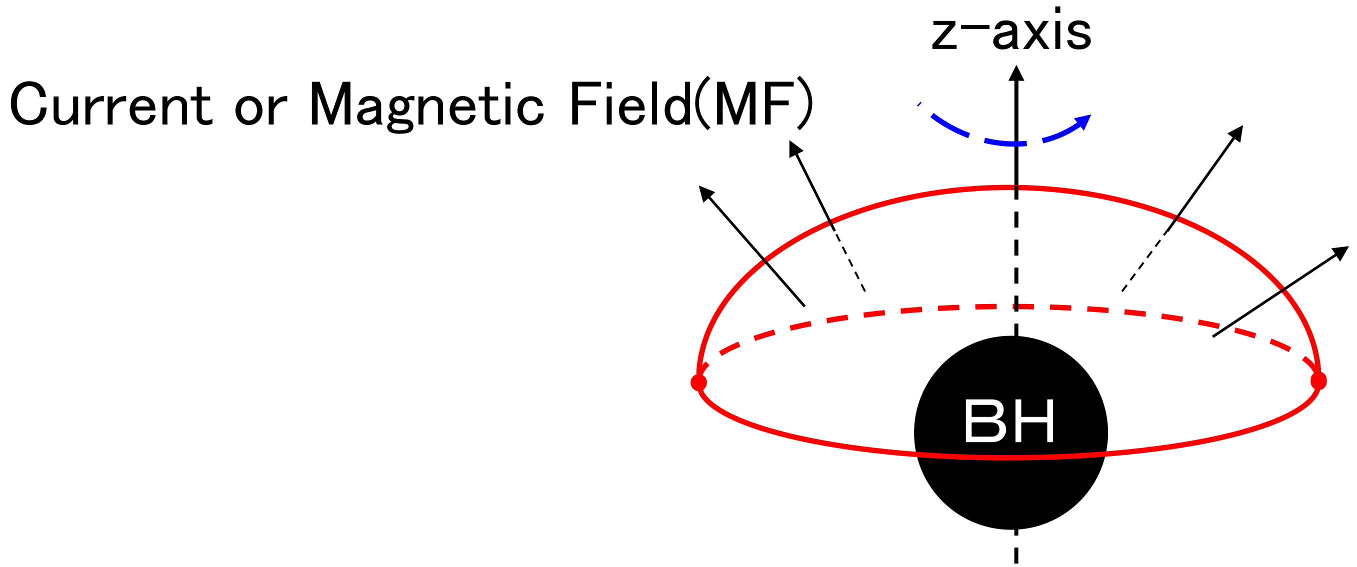
Stationary force-free electromagnetic field

Basic quantities

A_ϕ: vector potential(magnetic flux)

$\Omega_\mathrm{F}(A_\phi)$: angular velocity of MF

$I(A_\phi)$: poloidal electric current



Grad–Shafraonv(GS) equation

$$\frac{\Delta}{\Sigma}\partial_r^2 A_\phi + \frac{\sin\theta}{\Sigma}\partial_\theta\left(\frac{\partial_\theta A_\phi}{\sin\theta}\right) + \frac{N}{D} = 0$$

$$D = \frac{\Sigma\Delta}{A} - \left(\Omega_{\mathrm{F}} - \frac{2aMr}{A}\right)^2 \frac{A\sin^2\theta}{\Sigma}$$

$$N = (\partial_i D)\partial^i A_\phi + \frac{\Sigma\sin^2\theta}{A}\left(\Omega_{\mathrm{F}} - \frac{2aMr}{A}\right)\frac{d\Omega_{\mathrm{F}}}{A_\phi}(\partial_i A_\phi)\partial^i A_\phi + I\frac{dI}{dA_\phi}$$

- quasi–linear elliptic type differential equation.
- The equation has singular surfaces:

  event horizon $\Delta = 0$

  light surface $D = 0$

Regularity condition

- Horizon boundary condition

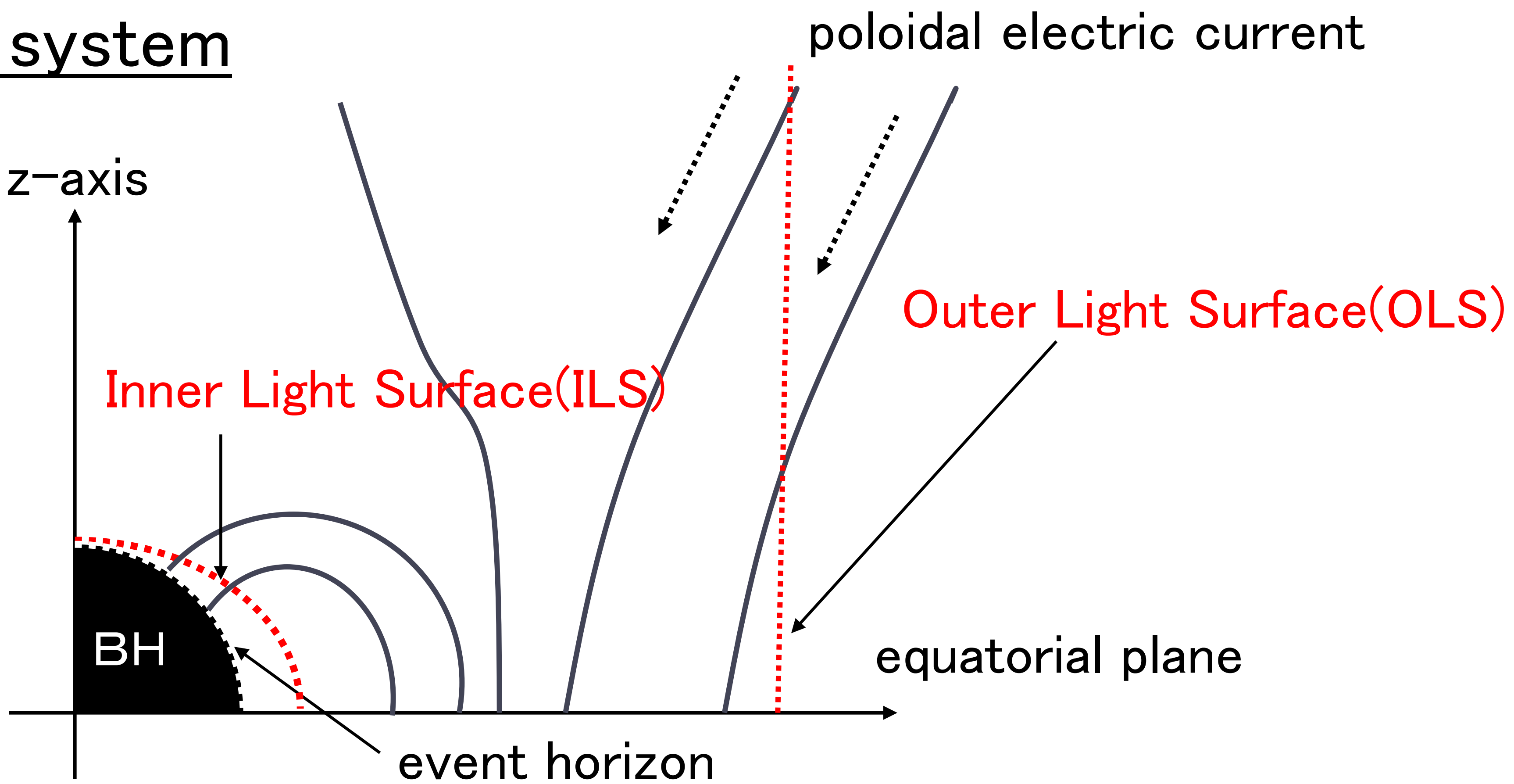
$$I = (\Omega_{\mathrm{H}} - \Omega_{\mathrm{F}}) \frac{(r_{\mathrm{H}}^2 + a^2)\sin\theta}{r_{\mathrm{H}}^2 + a^2\cos^2\theta} \partial_\theta A_\phi \quad \text{at} \quad r = r_{\mathrm{H}}$$

- Light surface condition

$$N = (\partial_i D)\partial^i A_\phi + \frac{\Sigma \sin^2\theta}{A}\left(\Omega_{\mathrm{F}} - \frac{2aMr}{A}\right)\frac{d\Omega_{\mathrm{F}}}{A_\phi}(\partial_i A_\phi)\partial^i A_\phi + I\frac{dI}{dA_\phi} = 0 \ \text{at} \ r = r_{\mathrm{LS}}$$

These equations give the boundary conditions at the event horizon and the light surface if one gives the electric current $I(A_\phi)$.

Stationary BH magnetosphere in the force–free system



We should search appropriate $I(A_\phi)$ to make the magnetic field lines smooth at three singular surfaces but it is difficult in general.

Slow rotation approximation

(Blandford & Znajek 1977)

We want to construct BH magnetospheres to analyze the BZ mechanism but it is difficult in general.

⇒ Let us consider slowly rotating BH and MF.

- $|a| << M$

  The lowest order expresses a magnetosphere in vacuum on the Schwarzschild background.

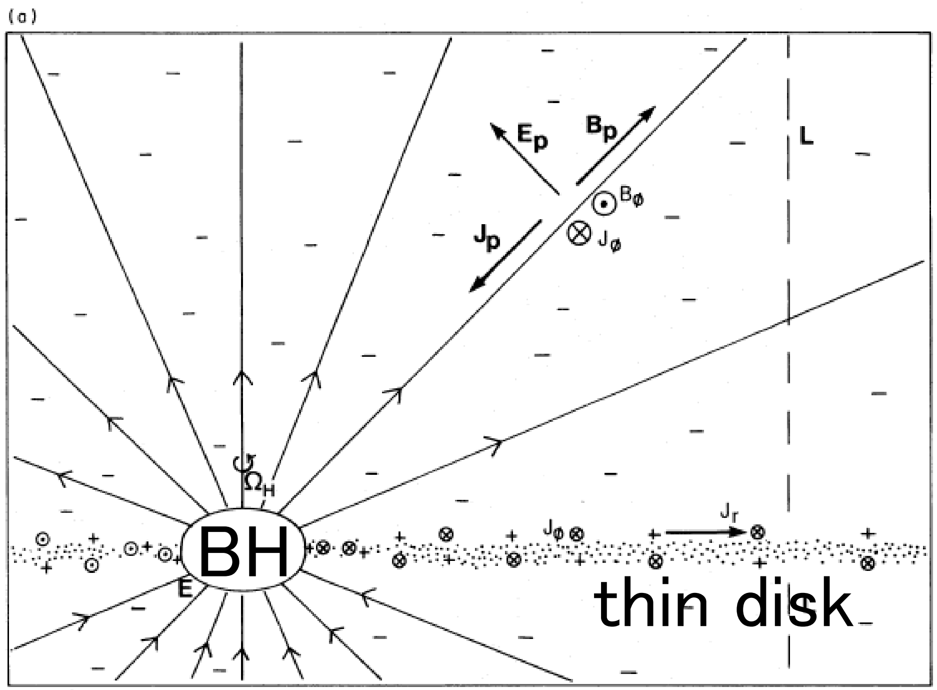
- $\Omega_\mathrm{F} \propto a$

  ILS and EH are degenerate. OLS goes to the infinity.

  Analysis of the GS eq. becomes more simpler.

Blandford-Znajek monopole solution

$$A_\phi = C(1 - \cos\theta) + (a/M)^2 f_{\mathrm{BZ}}(r) \sin^2\theta \cos\theta + \cdots$$



The BZ power

$$P_{\mathrm{BZ}} = \frac{C^2\pi}{24} \frac{a^2}{M^2}$$

Toward the BZ mechanism for a rapidly rotating BH

- Tanabe & Nagataki (2008)

  Calculation of the 4th order solution for the BZ monopole sol.

  $$P_{\rm BZ} = \frac{C^2\pi}{24}\frac{a^2}{M^2} + \frac{\pi(56-3\pi^2)C^2}{1050}\frac{a^4}{M^4}$$

  Better than 2nd order but not good for a rapidly rotating BH.

- Tchekhovskoy, Narayan, & McKinney (2010)

  Expansion with $\Omega_{\rm H}$. (phenomenological approach)

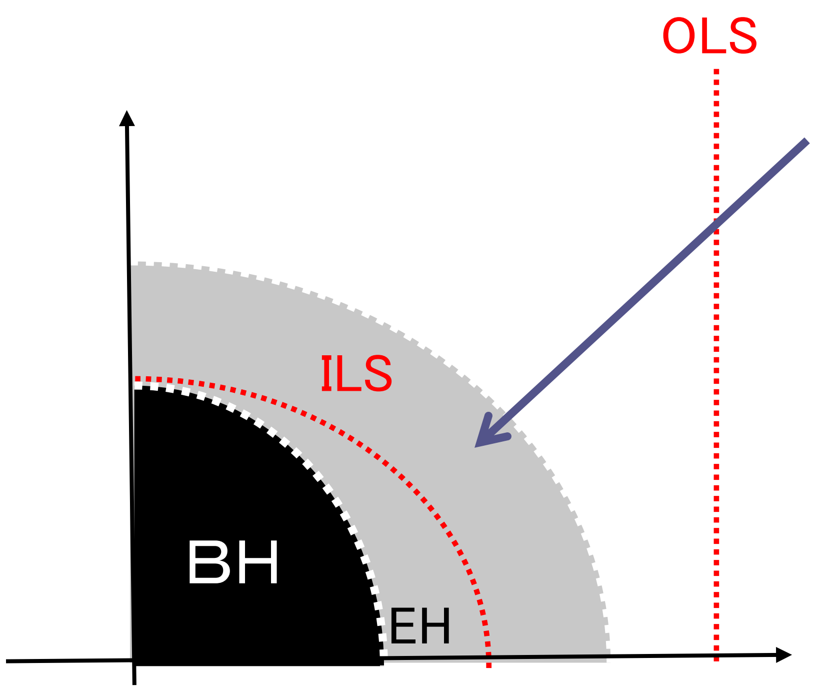
  $$P_{\rm BZ} = (2C^2\pi/3)((M\Omega_{\rm H})^2 + \alpha(M\Omega_{\rm H})^4 + \beta(M\Omega_{\rm H})^6) \quad \alpha \sim 1.38,\ \beta \sim -9.2$$

  This formula can express the BZ power obtained by MHD simulation for a rapidly rotating BH more precisely but why?

We want to analyze the BZ mechanism for a rapidly rotating BH analytically.

Perturbative analysis of the BZ power with a co-rotating magnetic field (YT 2013, in prep.)

Consider a magnetic field with $\Omega_F \sim \Omega_H$.



ILS is located near the EH.

⇒ Solve the GS eq. with the ILS condition in this region approximately.
It is sufficient to obtain the BZ power.

We do not restrict the spin parameter of BH!

Almost co-rotating magnetic field

$$\Omega_{\mathrm{F}} = \mathrm{const.} = (1-\epsilon)\Omega_{\mathrm{H}} \quad (0 < \epsilon \ll 1)$$

We expand all quantities as a power series of ϵ as follows

$$A_\phi = A_{\phi(0)} + \epsilon A_{\phi(1)} + \cdots$$

$$I = \epsilon I_{(1)} + \cdots$$

and so on.

Horizon boundary condition

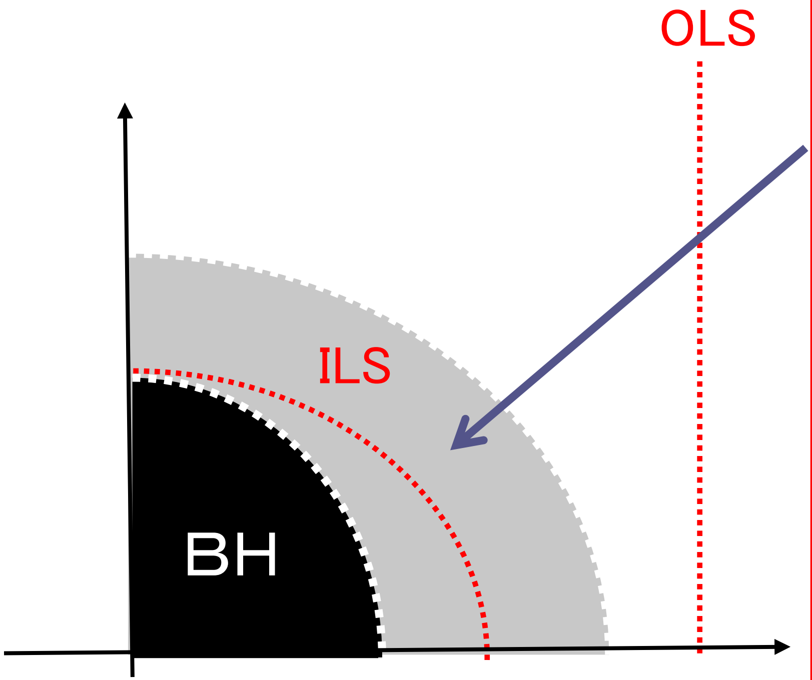
$$\left(I = (\Omega_{\mathrm{H}} - \Omega_{\mathrm{F}}) \frac{(r_{\mathrm{H}}^2 + a^2)\sin\theta}{r_{\mathrm{H}}^2 + a^2\cos^2\theta} \partial_\theta A_\phi \right)$$

Taylor expansion around the EH

Recall that we focus on a region near the EH.

A_ϕ is expressed as a Taylor series around $r = r_{\mathrm{H}}$:

$$A_\phi = A_{\phi(0)}|_{\mathrm{H}} + \partial_r A_{\phi(0)}|_{\mathrm{H}}(r - r_{\mathrm{H}}) + \cdots$$
$$+\epsilon\left(A_{\phi(1)}|_{\mathrm{H}} + \partial_r A_{\phi(1)}|_{\mathrm{H}}(r - r_{\mathrm{H}}) + \cdots\right) + \cdots$$



The ILS can express as

$$r_{\mathrm{ILS}} = r_{\mathrm{H}} + \epsilon^2 r_{\mathrm{ILS}(2)} + \cdots$$

Therefore, the ILS condition also expand as a Taylor series around $r = r_{\mathrm{H}}$.

Moreover, it can be written as a power series of ϵ.

Lowest order equations

$A_{\phi(0)}|_{\rm H}$, $\partial_r A_{\phi(0)}|_{\rm H}$, $I_{(1)}$

- GS eq. at the EH with $\Omega_{\rm F} = \Omega_{\rm H}$ (ϵ^0)

$$\sin\theta\partial_\theta\left(\frac{\partial_\theta A_{\phi(0)}|_{\rm H}}{\sin\theta}\right)+2(r_{\rm H}-M)\partial_r A_{\phi(0)}|_{\rm H}+\frac{(\partial_\theta\Sigma|_{\rm H})\partial_\theta A_{\phi(0)}|_{\rm H}}{\Sigma|_{\rm H}}=0$$

- Horizon boundary condition(HBC) (ϵ^1)

$$I_{(1)}=\frac{(r_{\rm H}^2+a^2)\sin\theta}{r_{\rm H}^2+a^2\cos^2\theta}\partial_\theta A_{\phi(0)}|_{\rm H}$$

- Inner light surface condition(ϵ^2)

$$\begin{aligned} D &= \frac{\Sigma\Delta}{A}-\left((1-\epsilon)\Omega_{\rm H}-\frac{2aMr}{A}\right)^2\frac{A\sin^2\theta}{\Sigma} \\ &= D_{(0)}+\epsilon D_{(1)}+\epsilon^2 D_{(2)} \end{aligned}$$

$$\frac{\partial_r\Delta}{\Sigma}r_{\rm ILS(2)}\partial_r D_{(0)}\partial_r A_{\phi(0)}$$
$$+\frac{1}{\Sigma}\left(\partial_\theta D_{(2)}+(\partial_r\partial_\theta D_{(0)}r_{\rm ILS(2)})\right)\partial_\theta A_{\phi(0)}+I_{(1)}\frac{dI_{(1)}}{dA_\phi}=0 \quad \text{at} \quad r=r_{\rm H}$$

where $r_{\rm ILS(2)}=-\frac{D_{(2)}|_{\rm H}}{\partial_r D_{(0)}|_{\rm H}}$.

Lowest order equations

$A_{\phi(0)}|_{\mathrm{H}}$, $\partial_r A_{\phi(0)}|_{\mathrm{H}}$, $I_{(1)}$

- GS eq. at the EH with $\Omega_{\mathrm{F}} = \Omega_{\mathrm{H}}$ (ϵ^0)

$$\sin\theta\partial_\theta\left(\frac{\partial_\theta A_{\phi(0)}|_{\mathrm{H}}}{\sin\theta}\right)+2(r_{\mathrm{H}}-M)\partial_r A_{\phi(0)}|_{\mathrm{H}}+\frac{(\partial_\theta\Sigma|_{\mathrm{H}})\partial_\theta A_{\phi(0)}|_{\mathrm{H}}}{\Sigma|_{\mathrm{H}}}=0$$

- Horizon boundary condition(HBC) (ϵ^1)

$$I_{(1)}=\frac{(r_{\mathrm{H}}^2+a^2)\sin\theta}{r_{\mathrm{H}}^2+a^2\cos^2\theta}\partial_\theta A_{\phi(0)}|_{\mathrm{H}}$$

We can find that solutions of the GS eq. and the HBC automatically satisfy the ILS condition!
Solving the GS eq. with some $\partial_r A_{\phi(0)}|_{\mathrm{H}}$, we obtain $A_{\phi(0)}|_{\mathrm{H}}$.
Then, $I_{(1)}$ is given by the HBC.

Estimation of the BZ power

$$P_{\mathrm{BZ}} = 2\pi \int_0^\pi d\theta \Omega_{\mathrm{F}} \cdot I|_{\mathrm{H}} \cdot \partial_\theta A_\phi|_{\mathrm{H}}$$

$$\Omega_{\mathrm{F}} = (1-\epsilon)\Omega_{\mathrm{H}}$$

$$A_\phi|_{\mathrm{H}} \sim A_{\phi(0)}|_{\mathrm{H}}$$

$$I|_{\mathrm{H}} \sim \epsilon I_{(1)} = \frac{(r_{\mathrm{H}}^2 + a^2)\sin\theta}{r_{\mathrm{H}}^2 + a^2\cos^2\theta}\partial_\theta A_{\phi(0)}|_{\mathrm{H}}$$

$$P_{\mathrm{BZ}} \sim 2\pi \int_0^\pi d\theta \epsilon (1-\epsilon)\Omega_{\mathrm{H}}^2 \frac{(r_{\mathrm{H}}^2 + a^2)\sin\theta}{r_{\mathrm{H}}^2 + a^2\cos^2\theta}(\partial_\theta A_{\phi(0)}|_{\mathrm{H}})^2$$

We can obtain a formula of the BZ power as we give $A_{\phi(0)}|_{\mathrm{H}}$ for $0 \le a \le M$.

Example (monopole configuration)

Let us assume that $\partial_r A_{\phi(0)}|_{\mathrm{H}} = 0$

$$\sin\theta\partial_\theta\left(\frac{\partial_\theta A_{\phi(0)}|_{\mathrm{H}}}{\sin\theta}\right) + \frac{(\partial_\theta\Sigma|_{\mathrm{H}})\partial_\theta A_{\phi(0)}|_{\mathrm{H}}}{\Sigma|_{\mathrm{H}}} = 0$$

$$\Rightarrow \quad A_{\phi(0)}|_{\mathrm{H}} = -\frac{C r_{\mathrm{H}}}{a}\tan^{-1}\left(\frac{a\cos\theta}{r_{\mathrm{H}}}\right) + D$$

- C and D are integration constants.
- This solution has a magnetic monopole charge.
- This solution reproduces the lowest order solution for the BZ monopole as we take the limit $a/M \to 0$.

$$A_{\phi(0)}|_{\mathrm{H}} \sim C\cos\theta + D \quad (a/M \ll 1)$$

The BZ power for a monopole configuration

$$P_{\rm BZ} \sim 2\pi \int_0^{\pi} d\theta \epsilon(1-\epsilon)\Omega_{\rm H}^2 \frac{(r_{\rm H}^2 + a^2)\sin\theta}{r_{\rm H}^2 + a^2\cos^2\theta}(\partial_\theta A_{\phi(0)}|_{\rm H})^2$$

Give as $\epsilon = 0.5$

It corresponds to the case of the BZ monopole solution.

$(\Omega_{\rm F} = 0.5\Omega_{\rm H})$

$$A_{\phi(0)}|_{\rm H} = -\frac{Cr_{\rm H}}{a}\tan^{-1}\left(\frac{a\cos\theta}{r_{\rm H}}\right) + D$$

Formula of the BZ power for a monopole case

$$P_{\mathrm{BZ}} = \frac{\pi C^2 M \Omega_{\mathrm{H}}^2}{r_{\mathrm{H}}} F(\omega_{\mathrm{H}})$$

$$r_{\mathrm{H}} = M^2 + \sqrt{M^2 + a^2}$$

$$\Omega_{\mathrm{H}} = \frac{a}{2M r_{\mathrm{H}}}$$

$$F(\omega_{\mathrm{H}}) = \frac{\omega_{\mathrm{H}} + 3\omega_{\mathrm{H}}^3 + (3\omega_{\mathrm{H}}^4 + 2\omega_{\mathrm{H}}^2 - 1)\tan^{-1}\omega_{\mathrm{H}}}{4\omega_{\mathrm{H}}^3 (1 + \omega_{\mathrm{H}}^2)}$$

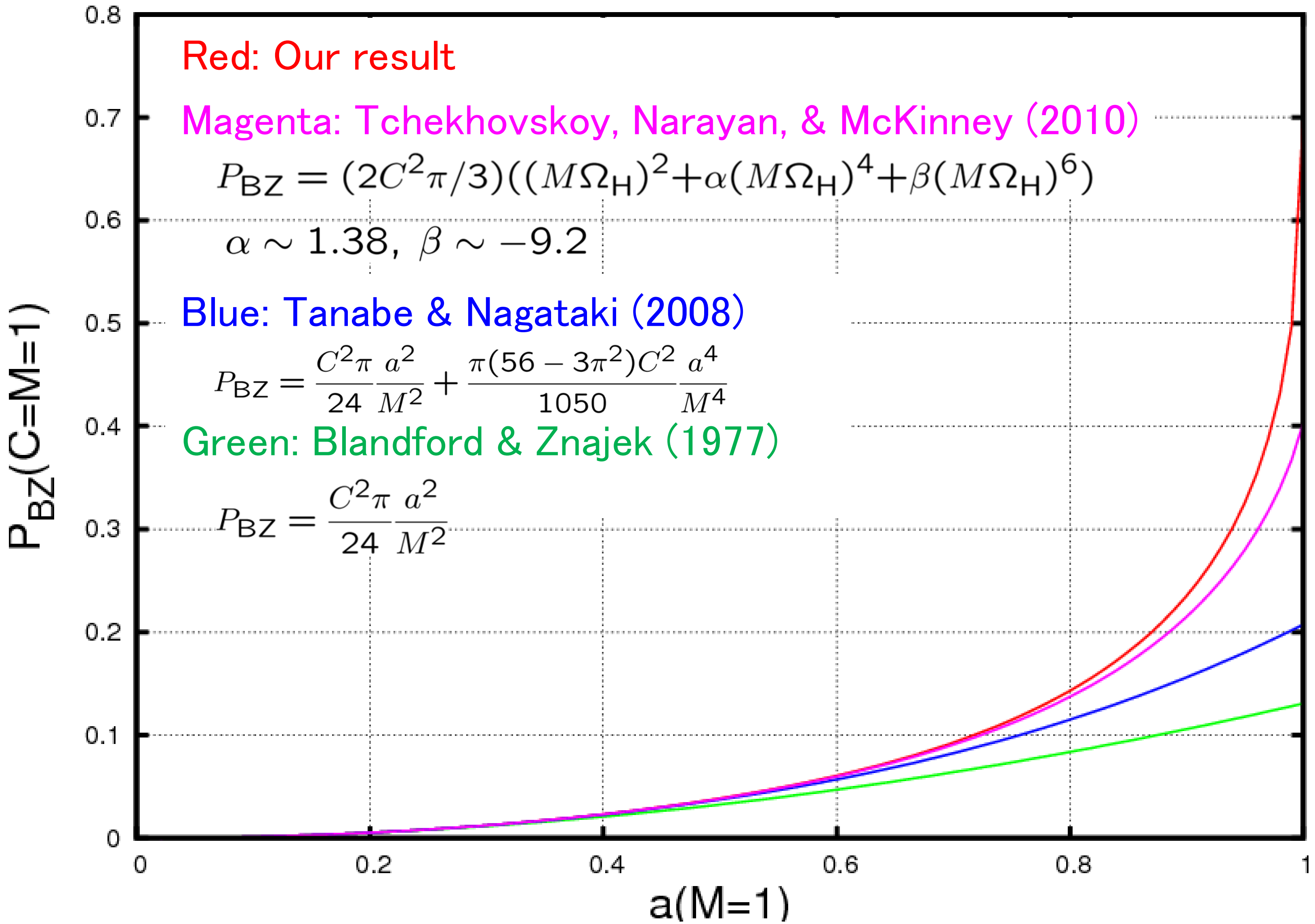
where $\omega_{\mathrm{H}} = 2M\Omega_{\mathrm{H}}$

Considering the case $a/M << 1$

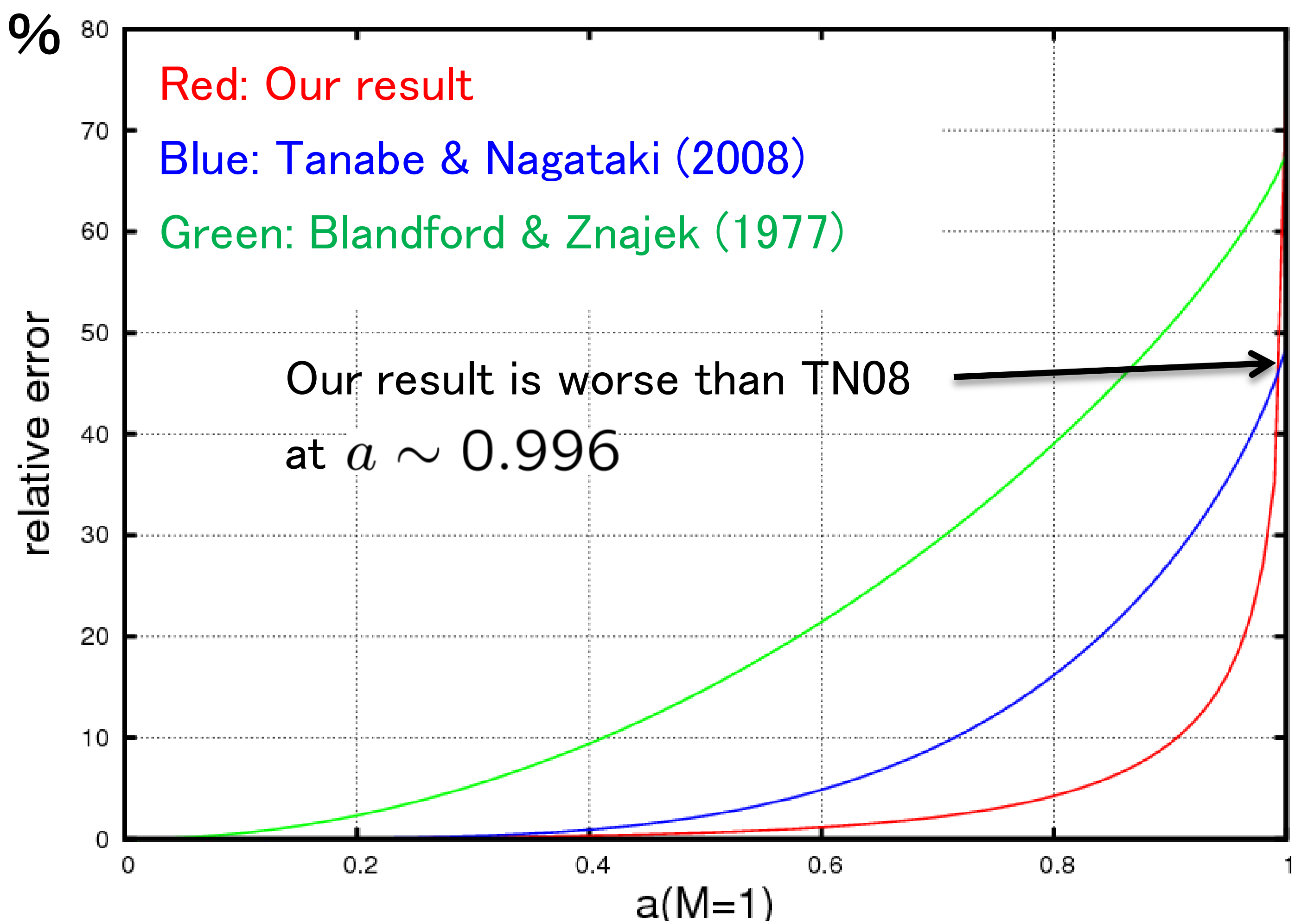
$$P_{\mathrm{BZ}} \sim \frac{\pi C^2}{24} \frac{a^2}{M^2}$$

Reproduce the BZ power of the BZ monopole solution!

Compare with the previous works



Relative error with the numerical result



Our result agrees well with the numerical result obtained by Tchekhovskoy, Narayan, & McKinney (2010) except $a \sim 1$.

Summary

- The Blandford–Znajek mechanism is expected as a source of a relativistic jet.
- We suggest a perturabative method to obtain the BZ power without restricting the spin parameter by focusing on an almost co–rotating magnetic field $\Omega_{\rm F} \sim \Omega_{\rm H}$.
- We obtain the formula of the BZ power for a monopole configuration by using the perturbative method for instance.
- The obtained formula agrees well with the numerical result obtained by Tchekhovskoy, Narayan, & McKinney (2010).
- The almost co–rotating approximation might not good near the extreme $a \sim 1$.